

FRAMES FOR SUBSPACES OF \mathbb{C}^N

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ABSTRACT. We present a theory of finite frames for subspaces of \mathbb{C}^N . The definition of a subspace frame is given and results analogous to those from frame theory for \mathbb{C}^N are proven.

1. INTRODUCTION

Frames have been used in many applications of signal processing. They give stable signal representations and allow modelling for noisy environments. Recent work with hyperspectral data has shown the need to consider subspaces of high dimensional spaces. In particular, ongoing work has led us to investigate frames for these subspaces.

2. FRAMES FOR \mathbb{C}^N

Let $\Phi = \{\varphi_j\}_{j=1}^s \subset \mathbb{C}^N$, where $s \geq N$. Φ is a *finite frame* for \mathbb{C}^N if there exist constants $A, B > 0$ such that

$$(1) \quad A\|f\|^2 \leq \sum_{j=1}^s |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in \mathbb{C}^N.$$

The numbers A, B are called the *frame bounds*. It is a well known fact that any spanning set is a frame for \mathbb{C}^N , while every frame is indeed a spanning set. A frame is *tight* if one can choose $A = B$ in the definition, i.e., if

$$(2) \quad \sum_{j=1}^s |\langle f, \varphi_j \rangle|^2 = A\|f\|^2, \quad \forall f \in \mathbb{C}^N.$$

Finally, a frame is *unit norm* if

$$(3) \quad \|\varphi_j\| = 1, \quad \forall j = 1, \dots, s.$$

If Φ satisfies (1), (2), and (3), then we say Φ is a *finite unit norm tight frame* (FUNTF) for \mathbb{C}^N . In this case, the frame bounds satisfy $A = B = s/N$. In particular, if Φ is a FUNTF with frame bounds $A = B = 1$, then Φ is an orthonormal basis.

Assume now that $\Phi = \{\varphi_j\}_{j=1}^s$ is a frame for \mathbb{C}^N . The *analysis operator* of Φ is defined as follows:

$$L : \mathbb{C}^N \rightarrow \mathbb{C}^s, \quad Lf := \{\langle f, \varphi_j \rangle\}_{j=1}^s.$$

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The *synthesis operator* is given by:

$$L^* : \mathbb{C}^s \rightarrow \mathbb{C}^N, \quad L^* \{c_j\}_{j=1}^s = \sum_{j=1}^s c_j \varphi_j.$$

One obtains the *frame operator* by composing L^* with L :

$$S : \mathbb{C}^N \rightarrow \mathbb{C}^N, \quad Sf = L^*Lf = \sum_{j=1}^s \langle f, \varphi_j \rangle \varphi_j$$

Some important properties of S are the following:

- (i) S is invertible and self-adjoint.
- (ii) Every $f \in \mathbb{C}^N$ can be represented as

$$(4) \quad f = \sum_{j=1}^s \langle f, S^{-1}\varphi_j \rangle \varphi_j = \sum_{j=1}^s \langle f, \varphi_j \rangle S^{-1}\varphi_j.$$

- (iii) Φ is a tight frame if and only if $S = AI$.

Based on equation (4), one defines the *dual frame* of Φ as $\tilde{\Phi} = \{\tilde{\varphi}_j\}_{j=1}^s := \{S^{-1}\varphi_j\}_{j=1}^s$; the frame operator of $\tilde{\Phi}$ is S^{-1} . If Φ is a tight frame for \mathbb{C}^N , then $S^{-1} = \frac{1}{A}I$, and the representation formula is simple:

$$f = \frac{1}{A} \sum_{j=1}^s \langle f, \varphi_j \rangle \varphi_j = \frac{1}{A} \sum_{j=1}^s \langle f, \varphi_j \rangle \varphi_j.$$

3. FRAMES FOR SUBSPACES OF \mathbb{C}^N

Let $\Phi = \{\varphi_j\}_{j=1}^s \subset \mathbb{C}^N$ and let W be a subspace of \mathbb{C}^N of dimension $r < N$. We say Φ is a *finite subspace frame* for W if $\text{span}(\Phi) = W$. It is clear from this definition that there exist constants $A, B > 0$ such that

$$(5) \quad A\|f\|^2 \leq \sum_{j=1}^s |\langle f, \varphi_j \rangle|^2 \leq B\|f\|^2, \quad \forall f \in W.$$

We note that if we had instead used (5) as our definition, then it would not necessarily imply that $\text{span}(\Phi) = W$ but rather that $\text{span}(\Phi) \supseteq W$. The unit norm property as well as the notion of a tight frame remain similar in this setting. More specifically, if we can take $A = B$ in (5) then we call Φ a *tight subspace frame*. Finally, if Φ is a finite unit norm tight subspace frame, then we say Φ is a *subspace FUNTF*.

We define L , L^* , and S exactly the same as in section 1, however we note that the properties of these maps change for subspace frames. In particular, we see:

- (a) $L : \mathbb{C}^N \rightarrow \mathbb{C}^s$ is no longer injective, but rather $\ker(L) = (\mathbb{C}^N \setminus W) \cup \{0\}$.
- (b) $L^* : \mathbb{C}^s \rightarrow \mathbb{C}^N$ is no longer surjective, but rather $\text{image}(L^*) = W$.
- (c) Based on (a) and (b), we see that $S : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is no longer invertible.

Because of (c), none of properties (i) - (iii) from section 1 hold for subspace frames. The question then becomes: in what sense do subspace frames satisfy properties (i) - (iii) above? Theorems below show that subspace frames satisfy natural modifications of the above properties.

Let W_{on} be a set of r orthonormal vectors such that $\text{span}(W_{on}) = W$. We will also consider W_{on} as an $N \times r$ matrix where the columns of this matrix are the vectors in the set W_{on} . We define Φ_W to be the $r \times s$ matrix whose columns are the coordinates of Φ in W_{on} ; that is:

$$(6) \quad \Phi_W := W_{on}^* \Phi,$$

where we have implicitly used the matrix form of Φ , that is the $N \times s$ matrix whose columns are the elements of Φ . The j^{th} column of Φ_W is the projected W -subspace coordinates of Φ .

Proposition 1. *The set Φ_W consisting of the columns of the matrix Φ_W is a frame for \mathbb{C}^r .*

Proof. Since $\text{span}(W_{on}) = W$, we have $\ker(W_{on}^*) \cap W = \{0\}$. Therefore, since $\text{span}(\Phi) = W$ as well, we see that $W_{on}^* \Phi$ has rank r . \square

We denote the analysis, synthesis, and frame operators of Φ_W by L_W , L_W^* , and S_W , respectively. In terms of the analysis operator, L , for Φ , $L_W = LW_{on}$. By proposition 1 we see that S_W will satisfy (i) - (iii).

Theorem 1. *Φ is a subspace FUNTF for W with frame bound A if and only if Φ_W is a FUNTF for \mathbb{C}^r with frame bound A .*

Proof. We do the forward direction first: let $g \in \mathbb{C}^r$, then:

$$\begin{aligned} \langle S_W g, g \rangle &= \langle L_W g, L_W g \rangle \\ &= \langle \Phi^* W_{on} g, \Phi^* W_{on} g \rangle \\ &= \sum_{j=1}^s |\langle W_{on} g, \varphi_j \rangle|^2 \\ &= A \|W_{on} g\|^2 \\ &= A \langle W_{on} g, W_{on} g \rangle \end{aligned}$$

Therefore we have:

$$\begin{aligned} \langle S_W g, g \rangle - A \langle W_{on} g, W_{on} g \rangle &= 0 \quad \implies \\ \langle S_W g, g \rangle - A \langle W_{on}^* W_{on} g, g \rangle &= 0 \quad \implies \\ \langle g, (S_W - AI)g \rangle &= 0 \quad \implies \\ S_W &= AI \end{aligned}$$

For the reverse direction, let $f \in W$. There exists $g \in \mathbb{C}^r$ such that $W_{on} g = f$. Therefore,

$$\begin{aligned} A \|f\|^2 &= A \langle f, f \rangle \\ &= A \langle W_{on} g, W_{on} g \rangle \\ &= \langle Ag, g \rangle \\ &= \langle S_W g, g \rangle \\ &= \langle W_{on}^* L^* L W_{on} g, g \rangle \\ &= \langle L W_{on} g, L W_{on} g \rangle \\ &= \langle Lf, Lf \rangle \\ &= \sum_{j=1}^s |\langle f, \varphi_j \rangle|^2 \end{aligned}$$

□

We define the dual frame of Φ_W in the usual way, that is $\tilde{\Phi}_W = S_W^{-1}\Phi_W$. We now define the *dual subspace frame* of Φ as follows:

$$(7) \quad \tilde{\Phi} := W_{on}\tilde{\Phi}_W = W_{on}S_W^{-1}W_{on}^*\Phi.$$

As the name implies, the set $\tilde{\Phi} = \{\tilde{\varphi}_j\}_{j=1}^s = \{W_{on}S_W^{-1}W_{on}^*\varphi_j\}_{j=1}^s$ will have the following properties:

Proposition 2. $\tilde{\Phi}$ is a subspace frame for W .

Proof. This follows from proposition 1. □

Theorem 2. Every $f \in W$ can be represented as

$$f = \sum_{j=1}^s \langle f, \tilde{\varphi}_j \rangle \varphi_j = \sum_{j=1}^s \langle f, \varphi_j \rangle \tilde{\varphi}_j.$$

Proof. The first representation formula is $\Phi\tilde{\Phi}^*f = f$ for all $f \in W$. Letting $f = W_{on}g$ for some $g \in \mathbb{C}^r$, we have:

$$(8) \quad \begin{aligned} \Phi\tilde{\Phi}^*f &= \Phi(W_{on}S_W^{-1}W_{on}^*\Phi)^*f \\ &= \Phi\Phi^*W_{on}(S_W^{-1})^*W_{on}^*(W_{on}g) \\ &= SW_{on}S_W^{-1}g \\ &= SW_{on}(W_{on}^*SW_{on})^{-1}g \end{aligned}$$

Since $W_{on}W_{on}^*$ is the identity on W ,

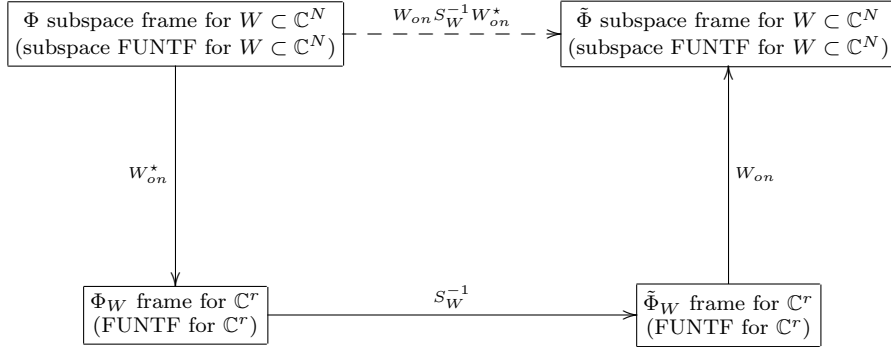
$$(8) \quad \begin{aligned} &= W_{on}W_{on}^*SW_{on}(W_{on}^*SW_{on})^{-1}g \\ &= W_{on}g \\ &= f \end{aligned}$$

The second representation formula is $\tilde{\Phi}\Phi^*f = f$ for all $f \in W$.

$$\begin{aligned} \tilde{\Phi}\Phi^*f &= (W_{on}S_W^{-1}W_{on}^*\Phi)\Phi^*f \\ &= W_{on}(W_{on}^*SW_{on})^{-1}W_{on}^*SW_{on}g \\ &= W_{on}g \\ &= f \end{aligned}$$

□

The following commutative diagram illustrates the above ideas:



4. FRAME POTENTIAL

Define the frame potential of a finite unit norm frame $\Phi = \{\varphi_j\}_{j=1}^s$ for \mathbb{C}^N as:

$$\text{FP}(\Phi) := \sum_{j=1}^s \sum_{k=1}^s |\langle \varphi_j, \varphi_k \rangle|^2.$$

In [1] a characterization of FUNTFs is given in terms of the frame potential:

Theorem 3 (Benedetto and Fickus 2002). *For a given N and s , let S^{N-1} denote the unit sphere in \mathbb{C}^N and consider:*

$$\text{FP} : \underbrace{S^{N-1} \times \dots \times S^{N-1}}_{s \text{ times}} \rightarrow [0, \infty).$$

Then:

- (1) *Every local minimizer of the frame potential is also a global minimizer.*
- (2) *If $s \leq N$, the minimum value of the frame potential is s , and the minimizers are precisely the orthonormal sequences in \mathbb{C}^N .*
- (3) *If $s \geq N$, the minimum value of the frame potential is s^2/N , and the minimizer are precisely the FUNTFs for \mathbb{C}^N .*

The following theorem is a trivial generalization of theorem 3:

Theorem 4. *For a given s and N , let W be a subspace of \mathbb{C}^N of dimension $r < N$ and consider the restricted frame potential:*

$$\text{FP}|_W : \underbrace{(S^{N-1} \times \dots \times S^{N-1})}_{s \text{ times}} \cap W \rightarrow [0, \infty).$$

Then:

- (1) *Every local minimizer of the restricted frame potential is also a global minimizer.*
- (2) *If $s \leq r$, the minimum value of the restricted frame potential is s , and the minimizers are precisely the orthonormal sequences in W .*
- (3) *If $s \geq r$, the minimum value of the restricted frame potential is s^2/r , and the minimizer are precisely the subspace FUNTFs for W .*

Theorem 4 shows that the minimum value of the frame potential depends on the dimension of the subspace W .

Proof. Let W_{on} be a set of r orthonormal vectors such that $\text{span}(W_{on}) = W$ and consider it as an $N \times r$ matrix. If $\Phi = \{\varphi_j\}_{j=1}^s$ is a finite unit norm set of vectors in W , then the coordinates of Φ in W_{on} are given by the $r \times s$ matrix $\Phi_W = W_{on}^* \Phi$. In [1] it is shown that $\text{FP}(\Phi) = \text{Tr}(S^2)$, where S is the frame operator of Φ . Using the previous two statements we then have:

$$\begin{aligned} \text{FP}|_W(\Phi) &= \text{Tr}(S^2) \\ &= \text{Tr}([(W_{on}\Phi_W)(W_{on}\Phi_W)^*]^2) \\ &= \text{Tr}([\Phi_W\Phi_W^*]^2) \\ &= \text{Tr}(S_W^2) \\ &= \text{FP}(\Phi_W) \end{aligned}$$

Since Φ_W is a finite unit norm set of vectors in \mathbb{C}^r , we can apply theorem 3 to get (1) and (2). Combining theorem 3 along with theorem 1 gives (3). \square

REFERENCES

- [1] John J. Benedetto and Matt Fickus. Finite normalized tight frames. *Adv. Comput. Math.*, 18:357–385, 2003.
- [2] J. Kovacevic and A Chebira. Life beyond bases: The advent of frames. *IEEE SP Mag.*, 2006. Feature article.

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