1 Vector Spaces

What is this course about?

1. Understanding the structural properties of a wide class of spaces which all share a similar additive and multiplicative structure
   structure = “vector addition and scalar multiplication” → vector spaces

2. The study of linear maps on finite dimensional vector spaces

We begin with vector spaces. First two examples:

1. \( \mathbb{R}^n = n\)-tuples of real numbers \( x = (x_1, \ldots, x_n) \), \( x_k \in \mathbb{R} \)
   vector addition: \( x+y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n) \)
   scalar multiplication: \( \lambda \in \mathbb{R}, \lambda x = \lambda (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n) \)

2. \( \mathbb{C}^n \) [on your own: review 1.A on complex numbers]

1. Definition of Vector Space

Scalars: Field \( \mathbb{F} \) (assume \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \) unless otherwise stated). So the previous two vector spaces can be written as \( \mathbb{F}^n \) with scalars \( \mathbb{F} \)

Let \( V \) be a set (for now).
**Definition 1** (Vector addition). \( u, v \in V \), assigns an element \( u + v \in V \)

**Definition 2** (Scalar multiplication). \( \lambda \in \mathbb{F}, \ v \in V \), assigns an element \( \lambda v \in V \)

**Definition 3** (Vector space). A set \( V \) is a vector space over the field \( \mathbb{F} \) if vector addition and scalar multiplication are defined, and the following properties hold \( (u, v, w \in V, \ a, b \in \mathbb{F}) \):

1. **Commutativity**: \( u + v = v + u \)
2. **Associativity**: \( (u + v) + w = u + (v + w) \) and \( (ab)v = a(bv) \)
3. **Additive Identity**: \( \exists 0 \in V \) such that \( v + 0 = v \)
4. **Additive Inverse**: for every \( v \) there exists \( w \) such that \( v + w = 0 \)
5. **Multiplicative Identity**: \( 1v = v \)
6. **Distributive Properties**: \( a(u + v) = au + av \) and \( (a + b)v = av + bv \)

If \( \mathbb{F} = \mathbb{R} \), “real vector space”
If \( \mathbb{F} = \mathbb{C} \), “complex vector space”

From here on out \( V \) will always denote a vector space.

Two more examples of vector spaces:

1. \( \mathbb{F}^\infty \): \( x = (x_1, x_2, \ldots) \) just like \( \mathbb{F}^n \)
2. \( \mathbb{F}^S \) = the set of functions \( f : S \to \mathbb{F} \) from \( S \) to \( \mathbb{F} \) [check on your own]

Now for some important properties...

**Proposition 1.** The additive identity is unique.

*Proof.* Let \( 0_1 \) and \( 0_2 \) be any two additive identities. Then

\[
0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2
\]

**Proposition 2.** The additive inverse is unique.
Proof. Let \( w_1 \) and \( w_2 \) be two additive inverses of \( v \). Then:

\[ w_1 = w_1 + 0 = w_1 + (v + w_2) = (v + w_1) + w_2 = 0 + w_2 = w_2 \]

Now we can write \(-v\) as the additive inverse of \( v \) and define subtraction as \( v - w = v + (-w) \). On the other hand, we still don’t “know” that \(-1v = -v\)!

**Notation:** We have \( 0_F \) and \( 0_V \). In the previous two propositions we dealt with \( 0_V \). Next we will handle \( 0_F \). We just write 0 for either and use the context to determine the meaning.

**Proposition 3.** \( 0_F v = 0_V \) for every \( v \in V \)

Proof.

\[ 0v = (0 + 0)v = 0v + 0v \implies 0v = 0 \]

Now the other way around...

**Proposition 4.** \( \lambda 0 = 0 \) for every \( \lambda \in F \)

**Proposition 5.** \( (-1)v = -v \) for all \( v \in V \)

Proof.

\[ v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0 \]

Now use uniqueness of additive inverse.

**End of Lecture 1**