Lecture Notes for Math 414: Linear Algebra II
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Beginning of Lecture 1

1 Vector Spaces

What is this course about?

1. Understanding the structural properties of a wide class of spaces which all share a similar additive and multiplicative structure
   structure = "vector addition and scalar multiplication" $\rightarrow$ vector spaces

2. The study of linear maps on finite dimensional vector spaces

We begin with vector spaces. First two examples:

1. $\mathbb{R}^n = n$-tuples of real numbers $x = (x_1, \ldots, x_n)$, $x_k \in \mathbb{R}$
   vector addition: $x+y = (x_1, \ldots, x_n)+(y_1, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n)$
   scalar multiplication: $\lambda \in \mathbb{R}$, $\lambda x = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$

2. $\mathbb{C}^n$ [on your own: review 1.A on complex numbers]

1.B Definition of Vector Space

Scalars: Field $\mathbb{F}$ (assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ unless otherwise stated). So the previous two vector spaces can be written as $\mathbb{F}^n$ with scalars $\mathbb{F}$

Let V be a set (for now).

Definition 1 (Vector addition)**.** $u, v \in V$, assigns an element $u + v \in V$

Definition 2 (Scalar multiplication)**.** $\lambda \in \mathbb{F}$, $v \in V$, assigns an element $\lambda v \in V$

Definition 3 (Vector space)**.** A set V is a vector space over the field $\mathbb{F}$ if vector addition and scalar multiplication are defined, and the following properties hold ($u, v, w \in V$, $a, b \in \mathbb{F}$):

1. Commutativity: $u + v = v + u$
2. Associativity: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$
3. Additive Identity: $\exists\, 0 \in V$ such that $v + 0 = v$
4. Additive Inverse: for every v there exists w such that $v + w = 0$
5. Multiplicative Identity: $1v = v$
6. Distributive Properties: $a(u + v) = au + av$ and $(a + b)v = av + bv$

If $\mathbb{F} = \mathbb{R}$, "real vector space"
If $\mathbb{F} = \mathbb{C}$, "complex vector space"

From here on out V will always denote a vector space

Two more examples of vector spaces:

1. $\mathbb{F}^\infty$: $x = (x_1, x_2, \ldots)$ just like $\mathbb{F}^n$
2. $\mathbb{F}^S$ = the set of functions $f : S \to \mathbb{F}$ from S to $\mathbb{F}$ [check on your own]

Now for some important properties...

Proposition 1. *The additive identity is unique.*

Proof. Let 0_1 and 0_2 be any two additive identities. Then

$$0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$$

□

Proposition 2. *The additive inverse is unique.*

Proof. Let w_1 and w_2 be two additive inverses of v. Then:

$$w_1 = w_1 + 0 = w_1 + (v + w_2) = (v + w_1) + w_2 = 0 + w_2 = w_2$$

□

Now we can write $-v$ as the additive inverse of v and define subtraction as $v - w = v + (-w)$. On the other hand, we still don't "know" that $-1v = -v$!

Notation: We have $0_{\mathbb{F}}$ and 0_V. In the previous two propositions we dealt with 0_V. Next we will handle $0_{\mathbb{F}}$. We just write 0 for either and use the context to determine the meaning.

Proposition 3. $0_{\mathbb{F}} v = 0_V$ *for every* $v \in V$

Proof.

$$0v = (0 + 0)v = 0v + 0v \implies 0v = 0$$

□

Now the other way around...

Proposition 4. $\lambda 0 = 0$ *for every* $\lambda \in \mathbb{F}$

Proposition 5. $(-1)v = -v$ *for all* $v \in V$

Proof.

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0$$

Now use uniqueness of additive inverse. □

End of Lecture 1