Beginning of Lecture 2

Warmup: Is the empty set $\emptyset$ a vector space?
Answer: No since $0 \notin \emptyset$

1.C Subspaces

A great way to find "new" vector spaces is to identify subsets of an existing vector space which are closed under addition and multiplication.

Definition 4 (Subspace)**.** $U \subset V$ is a subspace of V if U is also a vector space (using the same vector addition and scalar multiplication as V).

Proposition 6. *$U \subset V$ is a subspace if and only if:*

1. $0 \in U$
2. $u, w \in U \Longrightarrow u + w \in U$
3. $\lambda \in \mathbb{F}$ *and* $u \in U \Longrightarrow \lambda u \in U$

Now we can introduce more interesting examples of vector spaces, many of which are subspaces of $\mathbb{F}^S$ for some set S [you should verify these are vector spaces]:

1. $\mathcal{P}(\mathbb{F}) = \{p : \mathbb{F} \to \mathbb{F} : p(z) = \underbrace{a_0 + a_1 z + \cdots a_m z^m}_{\deg(p)=m}, \;\; a_k \in \mathbb{F} \;\; \forall\, k, m \in \mathbb{N}\}$
2. $C(\mathbb{R}; \mathbb{R})$ = real valued continuous functions
3. $C^m(\mathbb{R}^n; \mathbb{R})$ = real valued functions with continuous partial derivatives up to order m
4. $\mathcal{R}([0, 1]) = \{f : [0, 1] \to \mathbb{R} : \int_0^1 f(x)\, dx < \infty\}$.
5. $\mathbb{F}^{m,n}$ = the set of all $m \times n$ matrices with entries in $\mathbb{F}$
6. $\mathcal{S} = \{x : [0, 1] \to \mathbb{R}^n : x'(t)$ is continuous and $x'(t) = Ax(t),$ where $A \in \mathbb{R}^{n,n}\}$

Another convenient way to get new vector spaces is to add subspaces together (this is like the union of two sets, but for vector spaces!).

Definition 5 (Sum of subsets)**.** Suppose $U_1, \dots, U_m \subset V$. Then:

$$U_1 + \cdots + U_m := \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}.$$

Proposition 7. *Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$.*

An example:

$$\begin{aligned}
U_1 &= \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\} \\
U_2 &= \{x \in \mathbb{R}^3 : x_3 = 0\} \\
U_1 + U_2 &= \{x \in \mathbb{R}^3 : x = y + z,\ y_1 + y_2 + y_3 = 0 \text{ and } z_3 = 0\} \\
U_1 + U_2 &= \{x \in \mathbb{R}^3 : x = a(-1,0,1) + b(1,-1,0) + c(1,0,0) + d(0,1,0)\} \qquad (1) \\
U_1 + U_2 &= \mathbb{R}^3
\end{aligned}$$

Note there is redundancy in (1). We will be especially interested in situations that avoid this redundancy, i.e., subspace summations $U_1 + \cdots + U_m$ when the representation $u_1 + \cdots + u_m$ is unique.

Definition 6 (Direct sum)**.** Suppose that $U_1, \dots, U_m$ are subspaces of V.

- $U_1 + \cdots + U_m$ is a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as $u_1 + \cdots + u_m$ where $u_k \in U_k$.
- If $U_1 + \cdots + U_m$ is a direct sum, then we denote it as $U_1 \oplus \cdots \oplus U_m$

Examples:

1. Let U_k be the subspace of $\mathbb{F}^n$ such that only the k^{th} coordinate is nonzero:

   $$U_k = \{(\underbrace{0, \dots, 0}_{k-1}, x, 0, \dots, 0) : x \in \mathbb{F}\}$$

   Then

   $$\mathbb{R}^n = U_1 \oplus \cdots \oplus U_n$$

2. Recall the previous example with redundancy. That is not a direct sum. We can change U_2 though to get a direct sum:

   $$\begin{aligned}
   U_1 &= \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\} \\
   U_2 &= \{x \in \mathbb{R}^3 : x_1 = x_2 = x_3\} \\
   \mathbb{R}^3 &= U_1 \oplus U_2
   \end{aligned}$$

Notice in the second example that $U_1 \cap U_2 = \{0\}$. This leads us to the following proposition.

Proposition 8. *Let U, W be subspaces of V. Then,*

$$V = U \oplus W \iff U \cap W = \{0\}$$

The first example makes it tempting to propose the same pairwise intersection property for any number of subspaces, but this is not true! [try to come up with an example, then see the book] Instead we have the following proposition, which we can use to prove Proposition 8.

Proposition 9. *Suppose $U_1, \ldots, U_m$ are subspaces of V. Then*

$$U_1 + \cdots + U_m \text{ is a direct sum } \iff$$
$$0 = u_1 + \cdots + u_m,\; u_k \in U_k,\; \underline{\text{only}} \text{ when } u_k = 0\ \forall\, k$$

Proof. The $\Rightarrow$ direction is clear.
For the $\Leftarrow$ direction, let $v \in U_1 + \cdots + U_m$ and suppose we have two representations:

$$v = u_1 + \cdots + u_m = w_1 + \cdots + w_m$$

Then

$$0 = (u_1 - w_1) + \cdots + (u_m - w_m)$$

Since $u_k - w_k \in U_k$, we must have $u_k = w_k$ for each k. □

[try to prove Proposition 8 on your own using Proposition 9, then see the book].

2 Finite Dimensional Vector Spaces

2.A Span and Linear Independence

We saw last time that summing subspaces gives rise to new vector spaces. Now we keep track of each of the vectors that generate these spaces.

Definition 7 (Linear combination)**.** w is a linear combination of the vectors $v_1, \ldots, v_m \in V$ if $\exists\, a_1, \ldots, a_m \in \mathbb{F}$ such that

$$w = a_1 v_1 + \cdots a_m v_m$$

Definition 8 (Span)**.** The span of $v_1, \dots, v_m \in V$ is

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots a_m v_m : a_k \in \mathbb{F} \;\forall\, k\}$$

Analogous to the sum of subspaces, we have the following result.

Proposition 10. $\text{span}(v_1, \dots, v_m)$ *is the smallest subspace of* V *containing* $v_1, \dots, v_m$.

Nomenclature: If $\text{span}(v_1, \dots, v_m) = V$ then we say that $v_1, \dots, v_m$ spans V.

Definition 9 (Finite dimensional vector space)**.** V is finite dimensional if there exists a finite number of vectors $v_1, \dots, v_m$ (a list) such that $\text{span}(v_1, \dots, v_m) = V$.

Definition 10 (Infinite dimensional vector space)**.** V is infinite dimensional if it is not finite dimensional.

End of Lecture 2