Beginning of Lecture 3

Warmup: Is this a vector space?

1. $\{f \in C((0,1);\mathbb{R}) : f(x) = x^{-p} \text{ for some } p > 0\}$
   Answer: No (all three properties fail)

2. $\{f \in C(\mathbb{R};\mathbb{R}) : f \text{ is periodic of period } \sigma\}$
   Answer: Yes (contains zero function, closed under addition and scalar multiplication)

Examples:

1. $\mathcal{P}(\mathbb{F})$ is infinite dimensional [see the proof in the book].

2. $\mathcal{P}_m(\mathbb{F}) = \{p \in \mathcal{P}(\mathbb{F}) : \deg(p) \le m\}$ is finite dimensional:

$$\operatorname{span}(1, z, z^2, \ldots, z^m) = \mathcal{P}_m(\mathbb{F})$$

3. $U = \{f \in C(\mathbb{R};\mathbb{R}) : f \text{ is periodic of period } n \text{ for some } n \in \mathbb{N}\}$
   U is infinite dimensional

   Proof. Let $\mathcal{L} = v_1, \ldots, v_m$ be an arbitrary list from U, so that each v_k has period $n_k \in \mathbb{N}$. If $\ell = \operatorname{lcm}(n_1, \ldots, n_m)$, then any linear combination from $\mathcal{L}$ will have period which is at most ℓ. Therefore if p is a prime number such that $p > \ell$, $\sin(\frac{2\pi}{p}x) \notin \mathcal{L}$, but $\sin(\frac{2\pi}{p}x) \in U$, and thus $\operatorname{span}(\mathcal{L}) \neq U$. Since $\mathcal{L}$ was arbitrary we can conclude that no finite list will span U. □

It will be *very useful* to record if a list of vectors $v_1, \ldots, v_m$ has no redundancy in its span, just as we isolated sums of subspaces with no redundancy by defining the direct sum.

Definition 11 (Linear independence)**.** $v_1, \ldots, v_m \in V$ are linearly independent if whenever $0 = a_1 v_1 + \cdots + a_m v_m$, then necessarily $a_1 = \cdots = a_m = 0$.

Definition 12 (Linear dependence)**.** $v_1, \ldots, v_m \in V$ are linearly dependent if $\exists\, a_1, \ldots a_m$ with at least one $a_k \neq 0$ and $0 = a_1 v_1 + \cdots + a_m v_m$.

The notions of linear independence and linear dependence are extremely important!

Examples:

1. $(1, 0, 0), (0, 1, 0)$ are linearly independent in $\mathbb{F}^3$
2. $1, z, \ldots, z^m$ are linearly independent in $\mathcal{P}(\mathbb{F})$ [Why? Use the fact that a polynomial of degree m has at most m distinct zeros]
3. Recall example from sum of subspaces:
   - $(-1, 0, 1), (1, -1, 0), (1, 0, 0), (0, 1, 0)$ are linearly dependent
   - $(-1, 0, 1), (1, -1, 0), (1, 1, 1)$ are linearly independent

The following is a very useful lemma...

Lemma 1 (Linear Dependence Lemma, LDL)**.** *If $v_1, \ldots, v_m \in V$ are linearly dependent and $v_1 \neq 0$, then $\exists\, k \in \{2, \ldots, m\}$ such that*

1. $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$

2. If the v_k is removed from $v_1, \ldots, v_m$ then the resulting span *is the same as the original.*

Proof. Let $\mathcal{L} = v_1, \ldots, v_m$. For #1, by definition of linear dependence $\exists\, a_1, \ldots, a_m$ not all zero such that $0 = a_1 v_1 + \cdots + a_m v_m$. Let $k \in \{2, \ldots, m\}$ be the largest index such that $a_k \neq 0$. Then:

$$v_k = -\frac{a_1}{a_k} v_1 - \cdots - \frac{a_{k-1}}{a_k} v_{k-1} \tag{2}$$

For #2, let $\mathcal{L}^* = \mathcal{L} \setminus \{v_k\}$. Since $\mathcal{L}^* \subset \mathcal{L}$, $\operatorname{span}(\mathcal{L}^*) \subset \operatorname{span}(\mathcal{L})$. Let $u \in \operatorname{span}(\mathcal{L})$. Then:

$$u = a_1 v_1 + a_{k-1} v_{k-1} + a_k v_k + a_{k+1} v_{k+1} + \cdots + a_m v_m$$

Substitute (2) in for v_k and the sum is now in terms of $\mathcal{L}^*$, i.e., $u \in \operatorname{span}(\mathcal{L}^*)$. Thus $\operatorname{span}(\mathcal{L}) \subset \operatorname{span}(\mathcal{L}^*)$. $\square$

Now for our first theorem.

Theorem 1. *If $V = \operatorname{span}(v_1, \ldots, v_n)$ and $w_1, \ldots, w_m$ are linearly independent in V, then $m \leq n$.*

Proof. We will use the two lists and make successive reductions and additions using Lemma 1.
Note: $w_1, \dots, w_m$ linearly indpendent $\Rightarrow w_k \neq 0 \;\forall\, k$ [why?]

Add & reduce: Since $V = \text{span}(v_1, \dots, v_n)$ and $w_1 \in V$, then $w_1, v_1, \dots, v_n$ are linearly dependent. So Lemma 1 says at least one of the v_k can be removed. Up to a relabeling, we may assume it is v_n. So $\text{span}(w_1, v_1, \dots, v_{n-1})$ is the same as $\text{span}(v_1, \dots, v_n)$.

Now we can repeat: $w_2 \in V = \text{span}(w_1, v_1, \dots, v_{n-1})$ so $w_2, w_1, v_1, \dots, v_{n-1}$ are linearly dependent. Use Lemma 1 again, which says that one of them can be removed. The question is which? If it is w_1, then $w_1 \in \text{span}(w_2)$, which is a contradiction; so it must be one of the $v_1, \dots, v_{n-1}$. Without loss of generality (WLOG), we may assume it is v_{n-1} and so $\text{span}(w_2, w_1, v_1, \dots, v_{n-2}) = \text{span}(w_2, v_1, \dots, v_{n-1}) = V$.

Keep repeating. At each stage one of the v_k must be removed, else Lemma 1 implies that $w_j \in \text{span}(w_1, \dots w_{j-1})$ which is a contradiction.

The process stops when either we run out of w's ($m \leq n$) or we run out of v's ($m > n$). If $m > n$, then $\text{span}(w_1, \dots, w_n) = V$ and $m > n$. Thus $w_m \notin \text{span}(w_1, \dots, w_n) = V$, but this is a contradiction since $w_k \in V \;\forall\, k$. □

Proposition 11. *If V is finite dimensional and U is a subspace of V, then U is finite dimensional.*

End of Lecture 3