Beginning of Lecture 4

2.B Bases

span+ linear independence = basis

Definition 13. $v_1, \ldots, v_n \in V$ is a basis of V if $\text{span}(v_1, \ldots, v_n) = V$ and $v_1, \ldots, v_n$ are linearly independent.

Proposition 12. *$v_1, \ldots, v_n \in V$ is a basis of V if and only if*
$\forall\ v \in V,\ \exists!\, a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots a_n v_n$$

The notion of a basis is extremely important because it allows us to define a coordinate system for our vector spaces!

Examples:

1. $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is the standard basis of $\mathbb{F}^n$.
2. $1, z, \ldots, z^m$ is the standard basis for $\mathcal{P}_m(\mathbb{F})$
3. Let $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ (with addition mod N) and let $V = \{f : \mathbb{Z}_N \to \mathbb{C}\}$. The standard (time side) basis for V is $\delta_0, \ldots, \delta_{N-1}$ where

$$\delta_k(n) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$$

Indeed,

$$f(n) = \sum_{k=0}^{N-1} f(k) \delta_k(n)$$

Fourier analysis tells us that another (frequency side) basis for V is $e_0, \ldots, e_{N-1}$ where

$$e_k(n) = \frac{1}{\sqrt{N}} e^{2\pi i k n / N}$$

and

$$f(n) = \sum_{k=0}^{N-1} a_k e_k(n)$$

with

$$a_k = \hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-2\pi i k n/N}$$

The coefficients a_k define the function $\hat{f}(k)$ which is the Fourier transform of f.

If $v_1, \ldots, v_n$ spans V, it should have enough vectors to make a basis. Indeed:

Proposition 13. *If $\mathcal{L} = v_1, \ldots, v_n$ spans V, then $\mathcal{L}$ can be reduced to a basis.*

Proof. If $\mathcal{L}$ is linearly independent, then we are done. So assume it is not. We will selectively throw away vectors using the LDL.

Step 1: If $v_1 = 0$ remove v_1
Step 2: If $v_2 \in \text{span}(v_1)$, remove v_2
Step k: If $v_k \in \text{span}(v_1, \ldots, v_{k-1})$, remove v_k

Stop at Step n, getting a new list $\mathcal{L}^* = w_1, \ldots, w_m$. We still have $\text{span}(\mathcal{L}^*) = V$ since we only discarded vectors that were in the span of other vectors. We also have the property:

$$w_k \notin \text{span}(w_1, \ldots, w_{k-1}), \quad \forall\, k > 1$$

Thus by the contrapositive of LDL, $\mathcal{L}^*$ is linearly independent, and hence a basis. □

Corollary 1. *If V is finite dimensional, it has a basis.*

We just removed stuff from a spanning set to get a basis. We can also add stuff to a linearly independent set to get a basis.

Proposition 14. *If $\mathcal{L} = u_1, \ldots, u_m \in V$ is linearly independent, then $\mathcal{L}$ can be extended to a basis.*

Proof. Let $w_1, \ldots, w_n$ be a basis of V. Thus

$$\mathcal{L}^* = u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Apply the procedure in the proof of Proposition 13, and note that none of the u's get deleted [why?]. □

Now we show that every subspace U has a complementary subspace W that together direct sum to V.

Proposition 15. *Suppose V is finite dimensional and that U is a subspace of V. Then there exists another subspace W such that*

$$V = U \oplus W$$

Proof. V finite dimensional $\Rightarrow$ U finite dimensional $\Rightarrow$ U has a basis $u_1, \dots, u_m$. By the previous proposition we can extend $u_1, \dots, u_m$ to a basis of V, say $\mathcal{L} = u_1, \dots, u_m, w_1, \dots, w_n$. We show that $W = \text{span}(w_1, \dots, w_n)$ is the answer.

We need to show: (1) $V = U + W$, and (2) $U \cap W = \{0\}$. Since $\mathcal{L}$ is a basis, for any $v \in V$ we have:

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{u \in U} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{w \in W} = u + w \in U + W$$

Now suppose that $v \in U \cap W$. Then

$$v = a_1u_1 + \cdots + a_mu_m = b_1w_1 + \cdots + b_nw_n$$

which implies

$$a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0$$

But $\mathcal{L}$ is linearly independent so $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. □

2.C Dimension

Since a basis gives a unique representation of each $v \in V$, we should be able to say that the number of vectors in basis is the dimension of V. But to do so, we need to make sure every basis of V has the same number of vectors. Indeed:

Theorem 2. *Any two bases of a finite dimensional vector space have the same length.*

Proof. Let $\mathcal{B}_1 = v_1, \dots, v_m$ and $\mathcal{B}_2 = w_1, \dots, w_n$ be two bases of V. Since $\mathcal{B}_1$ is linearly independent and $\mathcal{B}_2$ spans V, $m \leq n$. Flipping the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$, we get $n \leq m$. □

Definition 14. The dimension of V is the length of $\mathcal{B}$ for any basis $\mathcal{B}$.

Proposition 16. *If U is a subspace of V, then* $\dim U \leq \dim V$

Examples:

1. $\dim \mathbb{F}^n = n$
   Remark: $\dim \mathbb{R}^2 = 2$ and $\dim \mathbb{C} = 1$, even though $\mathbb{R}^2$ can be identified with $\mathbb{C}$. The scalar field $\mathbb{F}$ cannot be ignored when computing the dimension of V!

2. $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$

Let $\mathcal{L} = v_1, \ldots, v_n$. If $\dim V = n$, then we need only check if $\mathcal{L}$ is linearly independent OR if $\text{span}(\mathcal{L}) = V$ to conclude that $\mathcal{L}$ is a basis for V.

Proposition 17. *Suppose* $\dim V = n$ *and let* $\mathcal{L} = v_1, \ldots, v_n$.

1. *If $\mathcal{L}$ is linearly independent, then $\mathcal{L}$ is a basis*

2. *If* $\text{span}(\mathcal{L}) = V$, *then $\mathcal{L}$ is a basis.*

Proof. Use Proposition 14 for (1) and Proposition 13 for (2). □

END OF LECTURE 4