BEGINNING OF LECTURE 4

2.B Bases

\[\text{span} + \text{linear independence} = \text{basis}\]

**Definition 13.** \(v_1, \ldots, v_n \in V\) is a basis of \(V\) if \(\text{span}(v_1, \ldots, v_n) = V\) and \(v_1, \ldots, v_n\) are linearly independent.

**Proposition 12.** \(v_1, \ldots, v_n \in V\) is a basis of \(V\) if and only if 
\[\forall \ v \in V, \ \exists! a_1, \ldots, a_n \in \mathbb{F} \text{ such that}\]
\[v = a_1 v_1 + \cdots + a_n v_n\]

The notion of a basis is **extremely important** because it allows us to define a **coordinate system** for our vector spaces!

Examples:

1. \((1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\) is the **standard basis** of \(\mathbb{F}^n\).

2. \(1, z, \ldots, z^m\) is the standard basis for \(\mathcal{P}_m(\mathbb{F})\)

3. Let \(\mathbb{Z}_N = \{0, 1, \ldots, N - 1\}\) (with addition mod \(N\)) and let \(V = \{f : \mathbb{Z}_N \rightarrow \mathbb{C}\}\). The standard (time side) basis for \(V\) is \(\delta_0, \ldots, \delta_{N-1}\) where

\[\delta_k(n) = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}\]

Indeed,

\[f(n) = \sum_{k=0}^{N-1} f(k) \delta_k(n)\]

Fourier analysis tells us that another (frequency side) basis for \(V\) is \(e_0, \ldots, e_{N-1}\) where

\[e_k(n) = \frac{1}{\sqrt{N}} e^{2\pi i kn/N}\]

and

\[f(n) = \sum_{k=0}^{N-1} a_k e_k(n)\]
with
\[ a_k = \hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)e^{-2\pi i kn/N} \]

The coefficients \(a_k\) define the function \(\hat{f}(k)\) which is the Fourier transform of \(f\).

If \(v_1, \ldots, v_n\) spans \(V\), it should have enough vectors to make a basis. Indeed:

**Proposition 13.** If \(\mathcal{L} = v_1, \ldots, v_n\) spans \(V\), then \(\mathcal{L}\) can be reduced to a basis.

**Proof.** If \(\mathcal{L}\) is linearly independent, then we are done. So assume it is not. We will selectively throw away vectors using the LDL.

Step 1: If \(v_1 = 0\) remove \(v_1\)
Step 2: If \(v_2 \in \text{span}(v_1)\), remove \(v_2\)
Step \(k\): If \(v_k \in \text{span}(v_1, \ldots, v_{k-1})\), remove \(v_k\)

Stop at Step \(n\), getting a new list \(\mathcal{L}^* = w_1, \ldots, w_m\). We still have \(\text{span}(\mathcal{L}^*) = V\) since we only discarded vectors that were in the span of other vectors. We also have the property:

\[ w_k \notin \text{span}(w_1, \ldots, w_{k-1}), \quad \forall k > 1 \]

Thus by the contrapositive of LDL, \(\mathcal{L}^*\) is linearly independent, and hence a basis.

**Corollary 1.** If \(V\) is finite dimensional, it has a basis.

We just removed stuff from a spanning set to get a basis. We can also add stuff to a linearly independent set to get a basis.

**Proposition 14.** If \(\mathcal{L} = u_1, \ldots, u_m \in V\) is linearly independent, then \(\mathcal{L}\) can be extended to a basis.

**Proof.** Let \(w_1, \ldots, w_n\) be a basis of \(V\). Thus

\[ \mathcal{L}^* = u_1, \ldots, u_m, w_1, \ldots, w_n \]

spans \(V\). Apply the procedure in the proof of Proposition 13, and note that none of the \(u\)'s get deleted [why?].
Now we show that every subspace $U$ has a complementary subspace $W$ that together direct sum to $V$.

**Proposition 15.** Suppose $V$ is finite dimensional and that $U$ is a subspace of $V$. Then there exists another subspace $W$ such that $V = U \oplus W$.

**Proof.** $V$ finite dimensional $\Rightarrow U$ finite dimensional $\Rightarrow U$ has a basis $u_1, \ldots, u_m$. By the previous proposition we can extend $u_1, \ldots, u_m$ to a basis of $V$, say $\mathcal{L} = u_1, \ldots, u_m, w_1, \ldots, w_n$. We show that $W = \text{span}(w_1, \ldots, w_n)$ is the answer.

We need to show: (1) $V = U + W$, and (2) $U \cap W = \{0\}$. Since $\mathcal{L}$ is a basis, for any $v \in V$ we have:

$$v = \sum_{u \in U} a_u u + \sum_{w \in W} b_w w = u + w \in U + W$$

Now suppose that $v \in U \cap W$. Then

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n$$

which implies

$$a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n = 0$$

But $\mathcal{L}$ is linearly independent so $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. $\square$

**2.C Dimension**

Since a basis gives a unique representation of each $v \in V$, we should be able to say that the number of vectors in basis is the dimension of $V$. But to do so, we need to make sure every basis of $V$ has the same number of vectors. Indeed:

**Theorem 2.** Any two bases of a finite dimensional vector space have the same length.

**Proof.** Let $\mathcal{B}_1 = v_1, \ldots, v_m$ and $\mathcal{B}_2 = w_1, \ldots, w_n$ be two bases of $V$. Since $\mathcal{B}_1$ is linearly independent and $\mathcal{B}_2$ spans $V$, $m \leq n$. Flipping the roles of $\mathcal{B}_1$ and $\mathcal{B}_2$, we get $n \leq m$. $\square$
Definition 14. The dimension of $V$ is the length of $B$ for any basis $B$.

Proposition 16. If $U$ is a subspace of $V$, then $\dim U \leq \dim V$

Examples:

1. $\dim \mathbb{F}^n = n$
   
   Remark: $\dim \mathbb{R}^2 = 2$ and $\dim \mathbb{C} = 1$, even though $\mathbb{R}^2$ can be identified with $\mathbb{C}$. The scalar field $\mathbb{F}$ cannot be ignored when computing the dimension of $V$!

2. $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$

Let $\mathcal{L} = v_1, \ldots, v_n$. If $\dim V = n$, then we need only check if $\mathcal{L}$ is linearly independent OR if $\text{span}(\mathcal{L}) = V$ to conclude that $\mathcal{L}$ is a basis for $V$.

Proposition 17. Suppose $\dim V = n$ and let $\mathcal{L} = v_1, \ldots, v_n$.

1. If $\mathcal{L}$ is linearly independent, then $\mathcal{L}$ is a basis

2. If $\text{span}(\mathcal{L}) = V$, then $\mathcal{L}$ is a basis.

Proof. Use Proposition 14 for (1) and Proposition 13 for (2).

End of Lecture 4