Theorem 3. \( \dim V < \infty, U_1 \) and \( U_2 \) subspaces of \( V \). Then

\[
\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)
\]

Proof. Proof will use 3 objects:

1. \( B = u_1, \ldots, u_m \) = basis of \( U_1 \cap U_2 \)
2. \( L_1 = v_1, \ldots, v_j \) = extension of \( B \) so that \( B \cup L_1 \) = basis for \( U_1 \)
3. \( L_2 = w_1, \ldots, w_k \) = extension of \( B \) so that \( B \cap L_2 \) = basis for \( U_2 \).

We will show that \( L = B \cup L_1 \cup L_2 \) is a basis for \( U_1 + U_2 \). This will complete the proof since if it is true, then

\[
\dim(U_1+U_2) = m+j+k = (m+j)+(m+k)-m = \dim U_1+\dim U_2-\dim(U_1\cap U_2)
\]

Clearly \( L \) spans \( U_1 + U_2 \) since \( \text{span}(L) \) contains both \( U_1 \) and \( U_2 \).

Now we show linear independence. Suppose:

\[
\sum_i a_i u_i + \sum_l b_l v_l + \sum_p c_p w_p = 0 \quad (3)
\]

Then:

\[
\sum_p c_p w_p = -\sum_i a_i u_i - \sum_l b_l v_l \in U_1
\]

But \( w_p \in U_2 \) by assumption, so

\[
\sum_p c_p w_p \in U_1 \cap U_2 \Rightarrow \sum_p c_p w_p = \sum_q d_q u_q \text{ for some } d_q
\]

Now, \( (u_1, \ldots, u_m, w_1, \ldots, w_k) \) is a basis for \( U_2 \). Thus:

\[
\sum_p c_p w_p - \sum_q d_q u_q = 0 \Rightarrow c_p = 0, d_q = 0, \forall p, q
\]

Therefore (3) reduces to

\[
\sum_i a_i u_i + \sum_l b_l v_l = 0
\]

Repeat the previous argument. \( \square \)
3 Linear Maps

$V, W$ always vector spaces.

3.A The Vector Space of Linear Maps

**Definition 15.** Let $V, W$ be vector spaces over the same field $F$. A function $T : V \to W$ is a linear map if it has the following two properties:

1. **additivity:** $T(u + v) = Tu + Tv, \ \forall u, v \in V$

2. **homogeneity:** $T(\lambda v) = \lambda(Tv) \ \forall \lambda \in F, v \in V$

The set of all linear maps from $V$ to $W$ is denoted $\mathcal{L}(V, W)$.

*Note:* You could say $T$ is linear if it “preserves the vector space structures of $V$ and $W$.”

Examples (read the ones in the book too!):

- Fix a point $x_0 \in \mathbb{R}$. **Evaluation at $x_0$** is a linear map:

$$T : C(\mathbb{R}; \mathbb{R}) \to \mathbb{R}$$

$$Tv = v(x_0)$$

- The **anti-derivative** is a linear map:

$$T : C(\mathbb{R}; \mathbb{R}) \to C^1(\mathbb{R}; \mathbb{R})$$

$$(Tv)(x) = \int_0^x v(y) \, dy$$

- Fix $b \in \mathbb{F}$. Define the forward shift operator as:

$$T : \mathbb{F}^\infty \to \mathbb{F}^\infty$$

$$T(v_1, v_2, v_3, \ldots) = (b, v_1, v_2, v_3, \ldots)$$

$T$ is a linear map if and only if $b = 0$ [why?].

Next we show that we can always find a linear map that takes whatever values we want on a basis, and furthermore, that it is completely determined by these values.
Theorem 4. Let $v_1, \ldots, v_n$ be a basis for $V$ and let $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_k = w_k, \quad \forall k$$

Proof. Define $T : V \to W$ as

$$T(a_1v_1 + \cdots + a_nv_n) = a_1w_1 + \cdots + a_nw_n$$

Clearly $Tv_k = w_k$ for all $k$. It is easy to see that $T$ is linear as well [see the book].

For uniqueness, let $S : V \to W$ be another linear map such that $Sv_k = w_k$ for all $k$. Then:

$$S(a_1v_1+ \cdots + a_nv_n) = \sum_{k=1}^{n} S(a_kv_k) = \sum_{k=1}^{n} a_kSv_k = \sum_{k=1}^{n} a_kw_k = T(a_1v_1+\cdots+a_nv_n)$$

The previous theorem is elementary, but highlights the fact that amongst all the maps from $V$ to $W$, linear maps are very special.

Theorem 5. $\mathcal{L}(V, W)$ is a vector space with the following vector addition and scalar multiplication operations:

- **vector addition:** $S, T \in \mathcal{L}(V, W)$, $(S + T)(v) = Sv + Tv \quad \forall v \in V$
- **scalar mult.:** $T \in \mathcal{L}(V, W), \lambda \in \mathbb{F}$, $(\lambda T)(v) = \lambda(Tv) \quad \forall v \in V$

Theorem 6. $\mathcal{L}(V, W)$ is finite dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Proof. Suppose $\dim V = n$ and $\dim W = m$ and let

- $\mathcal{B}_V = v_1, \ldots, v_n$
- $\mathcal{B}_W = w_1, \ldots, w_m$

be bases for $V$ and $W$ respectively. Define the linear transform $E_{p,q} : V \to W$ as

$$E_{p,q}(v_k) = \begin{cases} 0 & k \neq q \\ w_p & k = q \end{cases}, \quad p = 1, \ldots, m, \quad q = 1, \ldots, n$$
By Theorem 4, this uniquely defines each $E_{p,q}$. We are going to show that these $mn$ transformations $\{E_{p,q}\}_{p,q}$ form a basis for $\mathcal{L}(V,W)$.

Let $T : V \rightarrow W$ be a linear map. For each $1 \leq k \leq n$, let $a_{1,k}, \ldots, a_{m,k}$ be the coordinates of $Tv_k$ in the basis $\mathcal{B}_W$:

$$Tv_k = \sum_{p=1}^{m} a_{p,k}w_p$$

To prove spanning, we wish to show that:

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{p,q}E_{p,q} \quad (4)$$

Let $S$ be the linear map on the right hand side of (4). Then for each $k$,

$$Sv_k = \sum_{p} \sum_{q} a_{p,q}E_{p,q}v_k$$

$$= \sum_{p} a_{p,k}w_p$$

$$= Tv_k$$

So $S = T$, and since $T$ was arbitrary, $\{E_{p,q}\}_{p,q}$ spans $\mathcal{L}(V,W)$.

To prove linear independence, suppose that

$$S = \sum_{p} \sum_{q} a_{p,q}E_{p,q} = 0$$

Then $Sv_k = 0$ for each $k$, so

$$\sum_{p} a_{p,k}w_p = 0, \quad \forall k$$

But $w_1, \ldots, w_m$ are linearly independent, so $a_{p,k} = 0$ for all $p$ and $k$. $\square$

End of Lecture 5