Beginning of Lecture 5

Theorem 3. *$\dim V < \infty$, U_1 and U_2 subspaces of V. Then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof. Proof will use 3 objects:

1. $\mathcal{B} = u_1, \ldots, u_m =$ basis of $U_1 \cap U_2$
2. $\mathcal{L}_1 = v_1, \ldots, v_j =$ extension of $\mathcal{B}$ so that $\mathcal{B} \cup \mathcal{L}_1 =$ basis for U_1
3. $\mathcal{L}_2 = w_1, \ldots, w_k =$ extension of $\mathcal{B}$ so that $\mathcal{B} \cap \mathcal{L}_2 =$ basis for U_2.

We will show that $\mathcal{L} = \mathcal{B} \cup \mathcal{L}_1 \cup \mathcal{L}_2$ is a basis for $U_1 + U_2$. This will complete the proof since if it is true, then

$$\dim(U_1+U_2) = m+j+k = (m+j)+(m+k)-m = \dim U_1+\dim U_2-\dim(U_1 \cap U_2)$$

Clearly $\mathcal{L}$ spans $U_1 + U_2$ since span($\mathcal{L}$) contains both U_1 and U_2.

Now we show linear independence. Suppose:

$$\sum_i a_i u_i + \sum_l b_l v_l + \sum_p c_p w_p = 0 \tag{3}$$

Then:

$$\sum_p c_p w_p = -\sum_i a_i u_i - \sum_l b_l v_l \in U_1$$

But $w_p \in U_2$ by assumption, so

$$\sum_p c_p w_p \in U_1 \cap U_2 \Rightarrow \sum_p c_p w_p = \sum_q d_q u_q \text{ for some } d_q$$

Now, $(u_1, \ldots, u_m, w_1, \ldots, w_k)$ is a basis for U_2. Thus:

$$\sum_p c_p w_p - \sum_q d_q u_q = 0 \Rightarrow c_p = 0, d_q = 0, \ \ \forall\, p, q$$

Therefore (3) reduces to

$$\sum_i a_i u_i + \sum_l b_l v_l = 0$$

Repeat the previous argument. □

3 Linear Maps

V, W always vector spaces.

3.A The Vector Space of Linear Maps

Definition 15. Let V, W be vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is a linear map if it has the following two properties:

1. additivity: $T(u + v) = Tu + Tv, \ \forall\, u, v \in V$
2. homogeneity: $T(\lambda v) = \lambda(Tv) \ \forall\, \lambda \in \mathbb{F}, v \in V$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

Note: You could say T is linear if it "preserves the vector space structures of V and W."

Examples (read the ones in the book too!):

- Fix a point $x_0 \in \mathbb{R}$. Evaluation at x_0 is a linear map:

$$\begin{aligned} &T : C(\mathbb{R}; \mathbb{R}) \to \mathbb{R} \\ &Tv = v(x_0) \end{aligned}$$

- The anti-derivative is a linear map:

$$\begin{aligned} &T : C(\mathbb{R}; \mathbb{R}) \to C^1(\mathbb{R}; \mathbb{R}) \\ &(Tv)(x) = \int_0^x v(y)\, dy \end{aligned}$$

- Fix $b \in \mathbb{F}$. Define the forward shift operator as:

$$\begin{aligned} &T : \mathbb{F}^\infty \to \mathbb{F}^\infty \\ &T(v_1, v_2, v_3, \ldots) = (b, v_1, v_2, v_3, \ldots) \end{aligned}$$

  T is a linear map if and only if $b = 0$ [why?].

Next we show that we can always find a linear map that takes whatever values we want on a basis, and furthermore, that it is completely determined by these values.

Theorem 4. *Let $v_1, \dots, v_n$ be a basis for V and let $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that*

$$Tv_k = w_k, \quad \forall\, k$$

Proof. Define $T : V \to W$ as

$$T(a_1 v_1 + \cdots a_n v_n) = a_1 w_1 + \cdots + a_n w_n$$

Clearly $Tv_k = w_k$ for all k. It is easy to see that T is linear as well [see the book].

For uniqueness, let $S : V \to W$ be another linear map such that $Sv_k = w_k$ for all k. Then:

$$S(a_1 v_1 + \cdots a_n v_n) = \sum_{k=1}^{n} S(a_k v_k) = \sum_{k=1}^{n} a_k S v_k = \sum_{k=1}^{n} a_k w_k = T(a_1 v_1 + \cdots + a_n v_n)$$

□

The previous theorem is elementary, but highlights the fact that amongst all the maps from V to W, linear maps are very special.

Theorem 5. *$\mathcal{L}(V, W)$ is a vector space with the following vector addition and scalar multiplication operations:*

- *vector addition:* $S, T \in \mathcal{L}(V, W)$, $(S + T)(v) = Sv + Tv \ \forall\, v \in V$
- *scalar mult.:* $T \in \mathcal{L}(V, W)$, $\lambda \in \mathbb{F}$, $(\lambda T)(v) = \lambda(Tv) \ \forall\, v \in V$

Theorem 6. *$\mathcal{L}(V, W)$ is finite dimensional and*

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Proof. Suppose $\dim V = n$ and $\dim W = m$ and let

$$\mathcal{B}_V = v_1, \dots, v_n$$
$$\mathcal{B}_W = w_1, \dots, w_m$$

be bases for V and W respectively. Define the linear transform $E_{p,q} : V \to W$ as

$$E_{p,q}(v_k) = \begin{cases} 0 & k \neq q \\ w_p & k = q \end{cases}, \quad p = 1, \dots, m, \ \ q = 1, \dots, n$$

By Theorem 4, this uniquely defines each $E_{p,q}$. We are going to show that these mn transformations $\{E_{p,q}\}_{p,q}$ form a basis for $\mathcal{L}(V, W)$.

Let $T : V \to W$ be a linear map. For each $1 \leq k \leq n$, let $a_{1,k}, \ldots, a_{m,k}$ be the coordinates of Tv_k in the basis $\mathcal{B}_W$:

$$Tv_k = \sum_{p=1}^{m} a_{p,k} w_p$$

To prove spanning, we wish to show that:

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{p,q} E_{p,q} \tag{4}$$

Let S be the linear map on the right hand side of (4). Then for each k,

$$\begin{aligned} Sv_k &= \sum_{p} \sum_{q} a_{p,q} E_{p,q} v_k \\ &= \sum_{p} a_{p,k} w_p \\ &= Tv_k \end{aligned}$$

So $S = T$, and since T was arbitrary, $\{E_{p,q}\}_{p,q}$ spans $\mathcal{L}(V, W)$.

To prove linear independence, suppose that

$$S = \sum_{p} \sum_{q} a_{p,q} E_{p,q} = 0$$

Then $Sv_k = 0$ for each k, so

$$\sum_{p} a_{p,k} w_p = 0, \quad \forall\, k$$

But $w_1, \ldots, w_m$ are linearly independent, so $a_{p,k} = 0$ for all p and k. □

End of Lecture 5