**Beginning of Lecture 6**

Warmup: Let $U, W$ be 5-dimensional subspaces of $\mathbb{R}^9$. Can $U \cap W = \{0\}$?

**Answer:** No. First note that $\dim\{0\} = 0$. Then, using Theorem 3 we have:

$$\dim \mathbb{R}^9 = 9 \geq \dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

$$= 10 - \dim(U_1 \cap U_2)$$

$$\Rightarrow \dim(U_1 \cap U_2) \geq 1$$

**Proposition 18.** If $T : V \rightarrow W$ is a linear map, then $T(0) = 0$.

**Proof.**

$$T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$$

Usually the product of a vector from one vector space with a vector from another vector space is not well defined. However, for some pairs of linear maps, it is useful to define their product.

**Definition 16.** If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is

$$(ST)(u) = S(Tu), \quad \forall u \in U$$

**Note:** You must make sure the range of $T$ is in the domain of $S$!

**Another note:** Multiplication of linear maps is not commutative! In other words, in general $ST \neq TS$.

### 3.B Null Spaces and Ranges

For a linear map $T$, the collection of vectors that get mapped to zero and the collection of those that do not are very important.

**Definition 17.** For $T \in \mathcal{L}(V, W)$, the null space of $T$, null $T$, is:

$$\text{null } T = \{v \in V : Tv = 0\}$$

See examples in the book.

**Proposition 19.** For $T \in \mathcal{L}(V, W)$, null $T$ is a subspace of $V$. 
Proof. Check if it contains zero, closed under addition, closed under scalar multiplication:

- $T(0) = 0$ so $0 \in \text{null } T$
- $u, v \in \text{null } T$, then $T(u + v) = Tu + Tv = 0 + 0 = 0$
- $u \in \text{null } T$, $\lambda \in \mathbb{F}$, then $T(\lambda u) = \lambda Tu = \lambda 0 = 0$

\[ \square \]

**Definition 18.** A function $T : V \to W$ is injective if $Tu = Tv$ implies $u = v$.

**Proposition 20.** Let $T \in \mathcal{L}(V,W)$. Then

$$T \text{ is injective } \iff \text{null } T = \{0\}$$

**Proof.** For the $\Rightarrow$ direction, we already know that $0 \in \text{null } T$. Thus $T(v) = 0 = T(0)$, but since $T$ is injective $v = 0$.

For the $\Leftarrow$ direction, we have:

$$Tu = Tv \Rightarrow T(u - v) = 0 \Rightarrow u - v = 0 \Rightarrow u = v$$

\[ \square \]

**Definition 19.** For $T : V \to W$, the range of $T$ is:

$$\text{range } T = \{Tv : v \in V\}$$

**Proposition 21.** If $T \in \mathcal{L}(V,W)$, then range $T$ is a subspace of $W$.

**Definition 20.** A function $T : V \to W$ is surjective if range $T = W$.

**Theorem 7** (Rank-Nullity Theorem). Suppose $V$ is finite dimensional and $T \in \mathcal{L}(V,W)$. Then range $T$ is finite dimensional and

$$\dim V = \dim(\text{null } T) + \dim(\text{range } T)$$

**Proof.** Let $u_1, \ldots, u_m$ be a basis for null $T$, and extend it to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of $V$. So we need to show that $\dim \text{range } T = n$. To do so we prove that $Tv_1, \ldots, Tv_n$ is a basis for range $T$. 

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Let $v \in V$ and write:

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

$$\Rightarrow Tv = b_1 Tv_1 + \cdots + b_n Tv_n$$

Thus span$(Tv_1, \ldots, Tv_n) = \text{range } T$

Now we show that $Tv_1, \ldots, Tv_n$ are linearly independent. Suppose

$$c_1 Tv_1 + \cdots + c_n Tv_n = 0$$

$$\Rightarrow T(c_1 v_1 + \cdots + c_n v_n) = 0$$

$$\Rightarrow c_1 v_1 + \cdots + c_n v_n \in \text{null } T$$

$$\Rightarrow c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$$

But $v_1, \ldots, v_n, u_1, \ldots, u_m$ are linearly independent, so $c_j = d_k = 0$ for all $j, k$. Thus $Tv_1, \ldots, Tv_n$ are linearly independent.

**Corollary 2.** Suppose $V, W$ are finite dimensional and let $T \in \mathcal{L}(V, W)$. Then:

1. If $\dim V > \dim W$ then $T$ is not injective.
2. If $\dim V < \dim W$ then $T$ is not surjective.

**Proof.** Use the Rank-Nullity Theorem:

1. $\dim \text{null } T = \dim V - \dim \text{range } T \geq \dim V - \dim W > 0$
2. $\dim \text{range } T = \dim V - \dim \text{null } T \leq \dim V < \dim W$

**End of Lecture 6**