Beginning of Lecture 6

Warmup: Let U, W be 5-dimensional subspaces of $\mathbb{R}^9$. Can $U \cap W = \{0\}$?
Answer: No. First note that $\dim\{0\} = 0$. Then, using Theorem 3 we have:

$$\begin{aligned} \dim \mathbb{R}^9 = 9 \geq \dim(U_1 + U_2) &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \\ &= 10 - \dim(U_1 \cap U_2) \\ &\Rightarrow \dim(U_1 \cap U_2) \geq 1 \end{aligned}$$

Proposition 18. *If $T : V \to W$ is a linear map, then $T(0) = 0$.*

Proof.

$$T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow T(0) = 0$$

□

Usually the product of a vector from one vector space with a vector from another vector space is not well defined. However, for some pairs of linear maps, it is useful to define their product.

Definition 16. If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is

$$(ST)(u) = S(Tu), \quad \forall\, u \in U$$

Note: You must make sure the range of T is in the domain of S!
Another note: Multiplication of linear maps is not commutative! In other words, in general $ST \neq TS$.

3.B Null Spaces and Ranges

For a linear map T, the collection of vectors that get mapped to zero and the collection of those that do not are very important.

Definition 17. For $T \in \mathcal{L}(V, W)$, the null space of T, $\operatorname{null} T$, is:

$$\operatorname{null} T = \{v \in V : Tv = 0\}$$

See examples in the book.

Proposition 19. *For $T \in \mathcal{L}(V, W)$, $\operatorname{null} T$ is a subspace of V.*

Proof. Check if it contains zero, closed under addition, closed under scalar multiplication:

- $T(0) = 0$ so $0 \in \operatorname{null} T$
- $u, v \in \operatorname{null} T$, then $T(u+v) = Tu + Tv = 0 + 0 = 0$
- $u \in \operatorname{null} T$, $\lambda \in \mathbb{F}$, then $T(\lambda u) = \lambda T u = \lambda 0 = 0$

□

Definition 18. A function $T : V \to W$ is injective if $Tu = Tv$ implies $u = v$.

Proposition 20. *Let $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ is injective} \iff \operatorname{null} T = \{0\}$$

Proof. For the $\Rightarrow$ direction, we already know that $0 \in \operatorname{null} T$. Thus $T(v) = 0 = T(0)$, but since T is injective $v = 0$.

For the $\Leftarrow$ direction, we have:

$$Tu = Tv \Rightarrow T(u - v) = 0 \Rightarrow u - v = 0 \Rightarrow u = v$$

□

Definition 19. For $T : V \to W$, the range of T is:

$$\operatorname{range} T = \{Tv : v \in V\}$$

Proposition 21. *If $T \in \mathcal{L}(V, W)$, then $\operatorname{range} T$ is a subspace of W.*

Definition 20. A function $T : V \to W$ is surjective if $\operatorname{range} T = W$.

Theorem 7 (Rank-Nullity Theorem)**.** *Suppose V is finite dimensional and $T \in \mathcal{L}(V, W)$. Then $\operatorname{range} T$ is finite dimensional and*

$$\dim V = \dim(\operatorname{null} T) + \dim(\operatorname{range} T)$$

Proof. Let $u_1, \ldots, u_m$ be a basis for $\operatorname{null} T$, and extend it to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. So we need to show that $\dim \operatorname{range} T = n$. To do so we prove that $Tv_1, \ldots, Tv_n$ is a basis for $\operatorname{range} T$.

Let $v \in V$ and write:

$$\begin{aligned} v &= a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n \\ \Rightarrow Tv &= b_1Tv_1 + \cdots + b_nTv_n \end{aligned}$$

Thus $\operatorname{span}(Tv_1, \ldots, Tv_n) = \operatorname{range} T$

Now we show that $Tv_1, \ldots, Tv_n$ are linearly independent. Suppose

$$\begin{aligned} c_1Tv_1 + \cdots + c_nTv_n &= 0 \\ \Rightarrow T(c_1v_1 + \cdots + c_nv_n) &= 0 \\ \Rightarrow c_1v_1 + \cdots + c_nv_n &\in \operatorname{null} T \\ \Rightarrow c_1v_1 + \cdots + c_nv_n &= d_1u_1 + \cdots + d_mu_m \end{aligned}$$

But $v_1, \ldots, v_n, u_1, \ldots, u_m$ are linearly independent, so $c_j = d_k = 0$ for all j, k. Thus $Tv_1, \ldots, Tv_n$ are linearly independent. □

Corollary 2. *Suppose V, W are finite dimensional and let $T \in \mathcal{L}(V, W)$. Then:*

1. *If $\dim V > \dim W$ then T is not injective.*
2. *If $\dim V < \dim W$ then T is not surjective.*

Proof. Use the Rank-Nullity Theorem:

1. $\dim \operatorname{null} T = \dim V - \dim \operatorname{range} T \geq \dim V - \dim W > 0$
2. $\dim \operatorname{range} T = \dim V - \dim \operatorname{null} T \leq \dim V < \dim W$

□

End of Lecture 6