Beginning of Lecture 7

Very important applications:

- Homogeneous systems of equations
  m equations and n unknowns:

$$\begin{aligned}\sum_{k=1}^{n} a_{1,k}x_k &= 0 \\ &\vdots \\ \sum_{k=1}^{n} a_{m,k}x_k &= 0\end{aligned} \tag{5}$$

  where $a_{j,k} \in \mathbb{F}$ and $x = (x_1, \dots, x_n) \in \mathbb{F}^n$.
  Can you solve all m equations simultaneously? Clearly $x = 0$ is a solution. Are there any others? Define $T : \mathbb{F}^n \to \mathbb{F}^m$:

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} a_{1,k}x_k, \dots, \sum_{k=1}^{n} a_{m,k}x_k \right) \tag{6}$$

  Note: $T(0) = 0$ is equivalent to saying 0 is a solution of (5). Furthermore,

$$\text{Nontrivial solutions exist for (5)} \iff \dim \operatorname{null} T > 0$$

  But by the Rank-Nullity Theorem:

$$\dim \operatorname{null} T > 0 \iff \dim \mathbb{F}^n - \dim \operatorname{range} T > 0$$

  Since $\dim \operatorname{range} T \leq m$,

$$\text{if } n > m \implies \text{Nontrivial solutions exist for (5)}$$

- Inhomogeneous systems of equations: Let $c_k \in \mathbb{F}$ and consider:

$$\begin{aligned}\sum_{k=1}^{n} a_{1,k}x_k &= c_1 \\ &\vdots \\ \sum_{k=1}^{n} a_{m,k}x_k &= c_m\end{aligned} \tag{7}$$

New question, can you say for all $c = (c_1, \ldots, c_m) \in \mathbb{F}^m$ there exists at least one solution to (7)? Using the same T as defined in (6), we have:

$$\begin{aligned}
\text{A solution exists for (6)} &\iff \forall c \in \mathbb{F}^m, \exists\, x \in \mathbb{F}^n \text{ s.t. } T(x) = c \\
&\iff \operatorname{range} T = \mathbb{F}^m \\
&\iff \dim \operatorname{range} T = m \\
&\iff \dim \mathbb{F}^n - \dim \operatorname{null} T = m \\
&\iff \dim \operatorname{null} T = n - m
\end{aligned}$$

Since $\dim \operatorname{null} T \geq 0$, if $n < m$ then certainly there exists $c \in \mathbb{F}^m$ such that no solution exists for (7).

3.C Matrices

Definition 21. Let $T \in \mathcal{L}(V, W)$ and let $\mathcal{B}_V = v_1, \ldots, v_n$ and $\mathcal{B}_W = w_1, \ldots, w_m$ be bases of V and W respectively. The matrix of T with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ is the $m \times n$ matrix $\mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)$ (or just $\mathcal{M}(T)$ when $\mathcal{B}_V$ and $\mathcal{B}_W$ are clear) with entries $A_{j,k}$ defined by:

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j, \quad \forall\, k = 1, \ldots, n$$

Note: Recall the proof of the fact that $\dim \mathcal{L}(V, W) = mn$. In that proof we were implicitly using the matrix representation of T.

Another note: Recall the idea that a basis $\mathcal{B}_V = v_1, \ldots, v_n$ for a vector space V gives *coordinates* for V. That is, for all $v \in V$, there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n$$

So the n-tuple $(a_1, \ldots, a_n) \in \mathbb{F}^n$ is a *coordinate representation* of the vector v in the basis $\mathcal{B}_V$. If we change the basis, say to $\mathcal{B}'_V$, we change the coordinate representation of v say to $(a'_1, \ldots, a'_n)$, but we *do not* change v.

Similarly, the matrix $\mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)$ can be thought of as a coordinate representation of the linear map $T \in \mathcal{L}(V, W)$ with respect to the bases $\mathcal{B}_V$ and $\mathcal{B}_W$. If we change the bases, we get a *new* matrix representation of T, but we *do not* change T; it is still the same linear map. [we will come back to this with an example later]

Definition 22. $\mathbb{F}^{m,n}$ is the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

Proposition 22. *$\mathbb{F}^{m,m}$ is a vector space with the standard matrix addition and scalar multiplication.*

Proposition 23. $\dim \mathbb{F}^{m,n} = mn$.

We will derive matrix multiplication from the desire that $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ for all S, T for which ST makes sense. Suppose $T : U \to V$, $S : V \to W$, and that $\mathcal{B}_V = \{v_r\}_{r=1}^n$ is basis for V, $\mathcal{B}_W = \{w_j\}_{j=1}^m$ is a basis for W, and $\mathcal{B}_U = \{u_k\}_{k=1}^p$ is a basis for U. Let $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$. Then for each $1 \leq k \leq p$:

$$\begin{aligned}
(ST)u_k &= S\left(\sum_{r=1} nC_{r,k}v_r\right) \\
&= \sum_{r=1}^{n} C_{r,k}Sv_r \\
&= \sum_{r=1}^{n} C_{r,k}\sum_{j=1}^{m} A_{j,r}w_j \\
&= \sum_{j=1}^{m}\left(\sum_{r=1}^{n} A_{j,r}C_{r,k}\right)w_j
\end{aligned}$$

Thus we define matrix multiplication as:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}$$

[read the rest of 3.C on matrix multiplication on your own]

End of Lecture 7