Beginning of Lecture 8

3.D Invertibility and Isomorphic Vector Spaces

Definition 23. A linear map that is both injective and surjective is called bijective.

Definition 24. A linear map $T \in \mathcal{L}(V, W)$ is invertible if $\exists\, S \in \mathcal{L}(W, V)$ such that $ST = I_V$ and $TS = I_W$. Such a map S is an inverse of T.

Proposition 24. *An invertible linear map has a unique inverse.*

Proof. Let S_1 and S_2 be two inverses of $T \in \mathcal{L}(V, W)$. Then:

$$S_1 = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = S_2$$

□

Notation: Thus we can denote the inverse of T as $T^{-1} \in \mathcal{L}(W, V)$.

Theorem 8.

$$T \in \mathcal{L}(V, W) \text{ is invertible} \Longleftrightarrow T \text{ is bijective}$$

Proof. For the $\Longrightarrow$ direction: Need to show T is injective and surjective.
Suppose:

$$Tv_1 = Tv_2 \Rightarrow T^{-1}Tv_1 = T^{-1}Tv_2 \Rightarrow v_1 = v_2$$

since $T^{-1}T = I$. Thus T is injective.
Now suppose $w \in W$. Then:

$$TT^{-1}w = w \Rightarrow T\underbrace{(T^{-1}w)}_{\in V} = w$$

and so T is surjective.

Now for the $\Longleftarrow$ direction: Need to show T is invertible. To do so we define a map $S \in \mathcal{L}(W, V)$ and show that $ST = I$ and $TS = I$.
Define $S : W \to V$ as:

$$Sw := \text{unique } v \in V \text{ s.t. } Tv = w \text{ (i.e., } Sw = v \Leftrightarrow Tv = w)$$

Note S is well defined only because T is bijective! By construction we have $TS = I$. To show that $ST = I$, let $v \in V$, then:

$$T(STv) = (TS)(Tv) = Tv \Rightarrow ST = I \text{ since } T \text{ is injective}$$

Now we need to show that $S \in \mathcal{L}(W, V)$. For additivity let $w_1, w_2 \in W$:

$$\begin{aligned} T(Sw_1 + Sw_2) &= TSw_1 + TSw_2 = w_1 + w_2 \\ &\Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2 \text{ by definition of } S \end{aligned}$$

For homogeneity use a similar argument:

$$T(\lambda Sw) = \lambda T(Sw) = \lambda w \Rightarrow S(\lambda w) = \lambda Sw$$

□

We now want to formalize the notion of when two vector spaces are essentially the same.

Definition 25. Two parts:

- An isomorphism is an invertible linear map (i.e., a bijection)
- V, W are isomorphic if there exists $T \in \mathcal{L}(V, W)$ such that T is an isomorphism. We write $V \cong W$.

Theorem 9.

$$V \cong W \iff \dim V = \dim W$$

Proof. For the $\Longrightarrow$ direction, we know then there is a bijection $T \in \mathcal{L}(V, W)$. Thus $\operatorname{null} T = \{0\}$ and $\operatorname{range} T = W$, so by Rank-Nullity Theorem:

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = 0 + \dim W = \dim W$$

For the $\Longleftarrow$ direction, let $v_1, \ldots, v_n$ be a basis for V and let $w_1, \ldots, w_n$ be a basis for W. Define $T : V \to W$ as:

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots c_nw_n$$

It is easy to see $T \in \mathcal{L}(V, W)$, T is injective, T is surjective. Thus T defines an isomorphism. □

Corollary 3. *If* $\dim V = n$*, then* $V \cong \mathbb{F}^n$.

Remark: This *proves* that we can think of the coordinates of any $v \in V$ in a basis $\mathcal{B}_V = v_1, \ldots, v_n$ as a unique representation in $\mathbb{F}^n$, with the vector space structure of V carried over to $\mathbb{F}^n$. Indeed, define the matrix of $v \in V$ with respect to the basis $\mathcal{B}_V$ as the $n \times 1$ matrix:

$$\mathcal{M}(v; \mathcal{B}_V) := \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where

$$v = c_1 v_1 + \cdots + c_n v_n$$

The linear map $\mathcal{M}(\cdot, \mathcal{B}_V) : V \to \mathbb{F}^n$ (note $\mathbb{F}^{n,1} \cong \mathbb{F}^n$ trivially) is an isomorphism.

Corollary 4. *If* $\dim V = n$ *and* $\dim W = m$*, then* $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$*.*

Proof. This follows easily since we already proved that $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$. □

Proposition 25. *Let* $\mathcal{B}_V = v_1, \ldots, v_n$ *be a basis of* V *and let* $\mathcal{B}_W = w_1, \ldots, w_m$ *be a basis of* W*. Then* $\mathcal{M}(\cdot; \mathcal{B}_V, \mathcal{B}_W) : \mathcal{L}(V, W) \to \mathbb{F}^{m,n}$ *is an isomorphism.*

Proposition 26. *Let* $T \in \mathcal{L}(V, W)$*, let* $v \in V$*, and let* $\mathcal{B}_V$ *and* $\mathcal{B}_W$ *be bases of* V *and* W *respectively. Then:*

$$\mathcal{M}(Tv; \mathcal{B}_W) = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)\mathcal{M}(v; \mathcal{B}_V)$$

[See the book for the proofs of the previous two propositions.]

Example: Let $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ be the differentiation operator, defined by $Dp = p'$. Let's compute the matrix $\mathcal{M}(D)$ of D with respect to the standard bases $\mathcal{B}_3 = 1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{B}_2 = 1, x, x^2$ of $\mathcal{P}_2(\mathbb{R})$. Since $Dx^n = (x^n)' = nx^{n-1}$ we have:

$$\mathcal{M}(D; \mathcal{B}_3, \mathcal{B}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

End of Lecture 8