Beginning of Lecture 9

Example: Let $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ be the differentiation operator, defined by $Dp = p'$. Let's compute the matrix $\mathcal{M}(D)$ of D with respect to the standard bases $\mathcal{B}_3 = 1, x, x^2, x^3$ of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{B}_2 = 1, x, x^2$ of $\mathcal{P}_2(\mathbb{R})$. Since $Dx^n = (x^n)' = nx^{n-1}$ we have:

$$\mathcal{M}(D; \mathcal{B}_3, \mathcal{B}_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Now lets consider a different basis for $\mathcal{P}_3(\mathbb{R})$, for example $\mathcal{B}_3' = 1 + x, x + x^2, x^2 + x^3, x^3$. Compute:

$$\begin{aligned} D(1+x) &= 1 \\ D(x+x^2) &= 1 + 2x \\ D(x^2+x^3) &= 2x + 3x^2 \\ D(x^3) &= 3x^2 \end{aligned}$$

Thus:

$$\mathcal{M}(D; \mathcal{B}_3', \mathcal{B}_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

Now consider the specific polynomial $p \in \mathcal{P}_3(\mathbb{R})$,

$$p(x) = 2 + x + 3x^2 + 5x^3 \Longrightarrow p'(x) = 1 + 6x + 15x^2$$

The coordinates of p in $\mathcal{B}_3$ and $\mathcal{B}_3'$, as well as p' in $\mathcal{B}_2$, are:

$$\mathcal{M}(p; \mathcal{B}_3) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 5 \end{pmatrix} \quad \mathcal{M}(p; \mathcal{B}_3') = \begin{pmatrix} 2 \\ -1 \\ 4 \\ 1 \end{pmatrix} \quad \mathcal{M}(p'; \mathcal{B}_2) = \begin{pmatrix} 1 \\ 6 \\ 15 \end{pmatrix}$$

Computing Dp in terms of matrix multiplication with respect to $\mathcal{B}_3$ and $\mathcal{B}_2$

we should get back $\mathcal{M}(p';\mathcal{B}_2)$; indeed:

$$\begin{aligned}\mathcal{M}(Dp;\mathcal{B}_2) &= \mathcal{M}(D;\mathcal{B}_3,\mathcal{B}_2)\mathcal{M}(p;\mathcal{B}_3)\\ &= \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}\begin{pmatrix}2\\1\\3\\5\end{pmatrix}\\ &= \begin{pmatrix}1\\6\\15\end{pmatrix}\\ &= \mathcal{M}(p';\mathcal{B}_2)\end{aligned}$$

We should also be able to compute Dp in terms of matrix multiplication but with respect to $\mathcal{B}_3'$ and $\mathcal{B}_2$ and still get back $\mathcal{M}(p';\mathcal{B}_2)$; indeed:

$$\begin{aligned}\mathcal{M}(Dp;\mathcal{B}_2) &= \mathcal{M}(D;\mathcal{B}_3',\mathcal{B}_2)\mathcal{M}(p;\mathcal{B}_3')\\ &= \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 2 & 2 & 0\\ 0 & 0 & 3 & 3\end{pmatrix}\begin{pmatrix}2\\-1\\4\\1\end{pmatrix}\\ &= \begin{pmatrix}1\\6\\15\end{pmatrix}\\ &= \mathcal{M}(p';\mathcal{B}_2)\end{aligned}$$

Remark: As we said earlier, the choice of bases determines the matrix representation $\mathcal{M}(T;\mathcal{B}_V,\mathcal{B}_W)$ of the linear map $T\in\mathcal{L}(V,W)$. Later on we will prove important results about the choice of the bases the give the "nicest" possible matrix representation of T.

Definition 26. A linear map $T\in\mathcal{L}(V,V) =: \mathcal{L}(V)$ is an operator.

Remark: For the matrix of an operator $T\in\mathcal{L}(V)$, we assume that we take the same basis $\mathcal{B}_V$ for both the domain V and the range V, and thus write it as $\mathcal{M}(T;\mathcal{B}_V) := \mathcal{M}(T;\mathcal{B}_V,\mathcal{B}_V)$. Furthermore, $\mathcal{M}(T;\mathcal{B}_V)\in\mathbb{F}^{n,n}$, where $\dim V = n$, and so we see that $\mathcal{M}(T;\mathcal{B}_V)$ is a square matrix.

Theorem 10. *Suppose V is finite dimensional and $T\in\mathcal{L}(V)$. Then the following are equivalent:*

1. *T is bijective (i.e., invertible)*
2. *T is surjective*
3. *T is injective*

Remark: Not true if V is infinite dimensional!

Proof. We prove this by proving that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.
Clearly $1 \Rightarrow 2$ so that part is done.
Now suppose T is surjective, i.e., $\operatorname{range} T = V$. Then by the Rank-Nullity Theorem:

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ \Rightarrow \dim V &= \dim \operatorname{null} T + \dim V \\ \Rightarrow \dim \operatorname{null} T &= 0 \\ \Rightarrow \operatorname{null} T &= \{0\} \\ &\Rightarrow T \text{ is injective} \end{aligned}$$

So that takes care of $2 \Rightarrow 3$.
Now suppose T is injective. Then $\operatorname{null} T = \{0\}$ and $\dim \operatorname{null} T = 0$. Once again use the Rank-Nullity Theorem:

$$\begin{aligned} \dim V &= \dim \operatorname{null} T + \dim \operatorname{range} T \\ \Rightarrow \dim V &= 0 + \dim \operatorname{range} T \\ \Rightarrow \operatorname{range} T &= V \end{aligned}$$

Thus T is surjective. Since we assumed it was injective, this means T is bijective and so we have $3 \Rightarrow 1$ and we are done. □

4 Polynomials

Read on your own!

5 Eigenvalues, Eigenvectors, and Invariant Subspaces

Extremely important subject matter that is the heart of Linear Algebra and is used all over mathematics, applied mathematics, data science, and more.

For example, consider a graph $G = (\mathcal{V}, \mathcal{E})$ consisting of vertices $\mathcal{V}$ and edges $\mathcal{E}$; for example see Figure 1. You can encode this graph with a 6×6 matrix

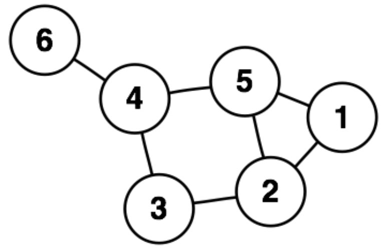

Figure 1: Graph with 6 vertices and 7 edges

L so that:

$$L_{j,k} = \begin{cases} \text{degree of vertex } k, & j = k \\ -1, & j \neq k \text{ and there is an edge between vertices } j \text{ and } k \\ 0, & \text{otherwise} \end{cases}$$

This matrix is called the graph Laplacian and it encodes connectivity properties of the graph through its eigenvalues and eigenvectors. If the nodes in the graph represent webpages, and the edges represent hyperlinks between the webpages, then a similar type of matrix represents the world wide web, and its eigenvectors and eigenvalues form the foundation of how Google computes search results!

5.A Invariant Subspaces

At the beginning of the course we defined a structure on sets V through the notion of a vector space. We then examined this structure further through subspaces, bases, and related notions. We then extended our study through linear maps between vector spaces, culminating in the Rank-Nullity Theorem and the notion of an isomorphism between two vector spaces with the same structure. Now we examine the structure of linear operators. The idea is that we will study the structure of $T \in \mathcal{L}(V)$ by finding nice structural decompositions of V relative to T.

Thought experiment: Let $T \in \mathcal{L}(V)$ and suppose

$$V = U_1 \oplus \cdots \oplus U_m$$

To understand T, we would need only understand $T_k = T|_{U_k}$ for each $k = 1, \ldots, m$. However, T_k may not be in $\mathcal{L}(U_k)$; indeed, T_k might map U_k to some other part of V. This is a problem, since we would like each restricted linear map T_k to be an operator itself on the subspace U_k. This leads us to the following definition.

Definition 27. Suppose $T \in \mathcal{L}(V)$. A subspace U of V is invariant under T if $Tu \in U$ for all $u \in U$, i.e., $T|_U \in \mathcal{L}(U)$.

Examples: $\{0\}$, V, $\operatorname{null} T$, $\operatorname{range} T$

Must an operator have any invariant subspaces other than $\{0\}$ and V? We will see... We begin with the study of one dimensional invariant subspaces.

End of Lecture 9