Beginning of Lecture 10

Definition 28. Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is an eigenvalue of T if there exists $v \in V$, $v \neq 0$, such that

$$Tv = \lambda v$$

Such a v is called an eigenvector of T.

Proposition 27. *$T \in \mathcal{L}(V)$ has a one dimensional invariant subspace if and only if T has an eigenvalue.*

Proof. First suppose that T has a one dimensional invariant subspace, which we denote as U. Since $\dim U = 1$, U must be of the form:

$$U = \{\lambda v : \lambda \in \mathbb{F}\} = \text{span}(v)$$

for some $v \in V$, $v \neq 0$. Since T is invariant under U, $Tv \in U$. Thus there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$.

Now suppose that T has an eigenvalue $\lambda \in \mathbb{F}$. Then there exists $v \in V$, $v \neq 0$, such that $Tv = \lambda v$. Then $U = \text{span}(v)$ is an invariant subspace under T. □

Proposition 28. *Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. The following are equivalent:*

1. *λ is eigenvalue of T*
2. *$T - \lambda I$ is not injective*
3. *$T - \lambda I$ is not surjective*
4. *$T - \lambda I$ is not invertible*

Example: The Laplacian for $V = \{f \in C^\infty([-\pi, \pi]; \mathbb{C}) : f(-\pi) = f(\pi)\}$ is defined as:

$$\Delta f = \frac{d^2 f}{dx^2}$$

The eigenvalues and eigenvectors of Δ are:

$$\lambda = -k^2, k \in \mathbb{Z}, \quad v(x) = e^{ikx} = \cos kx + i \sin kx$$

Notice the similarity between the eigenvectors of Δ and the Fourier Transform defined earlier on $\mathbb{Z}_N$...

Theorem 11. *Let $T \in \mathcal{L}(V)$. If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors, then $v_1, \ldots, v_m$ are linearly independent.*

Proof. Proof by contradiction. Suppose $v_1, \ldots, v_m$ are linearly dependent. Using the LDL, let k be the smallest index such that

$$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1}) \tag{8}$$

Thus

$$\begin{aligned} v_k &= a_1 v_1 + \cdots + a_{k-1} v_{k-1} \\ \Rightarrow T v_k &= a_1 T v_1 + \cdots + a_{k-1} T v_{k-1} \\ \Rightarrow \lambda_k v_k &= a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1} \end{aligned}$$

We also can conclude:

$$\begin{aligned} v_k &= a_1 v_1 + \cdots + a_{k-1} v_{k-1} \\ \Rightarrow \lambda_k v_k &= a_1 \lambda_k v_1 + \cdots + a_{k-1} \lambda_k v_{k-1} \end{aligned}$$

Combining the two expansions of $\lambda_k v_k$ yields:

$$0 = a_1(\lambda_k - \lambda_1) v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1}$$

Since k is the smallest index satisfying (8), $v_1, \ldots, v_{k-1}$ must be linearly independent. Thus $a_1 = \cdots = a_{k-1} = 0$ since $\lambda_k - \lambda_j \neq 0$ for all $k \neq j$. But then $v_k = 0$, which is a contradiction. □

Corollary 5. *Suppose V is finite dimensional. Then $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvalues.*

End of Lecture 10