Beginning of Lecture 11

5.B Eigenvectors and Upper-Triangular Matrices

One of the main differences between operators and general linear maps is that we can take powers of operators! This will lead to many interesting results...

Definition 29. Let $T \in \mathcal{L}(V)$ and let $m \in \mathbb{Z}$, $m > 0$.

- $T^m = T \cdots T$ (composition m times)
- $T^0 = I$
- If T is invertible, then $T^{-m} = (T^{-1})^m$

Definition 30. Suppose $T \in \mathcal{L}(V)$ and let $p \in \mathcal{P}(\mathbb{F})$ be given by:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

Then $p(T) \in \mathcal{L}(V)$ is defined as:

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

Theorem 12. *Let $V \neq \{0\}$ be a finite dimensional vector space over $\mathbb{C}$. Then every $T \in \mathcal{L}(V)$ has an eigenvalue.*

Proof. Suppose $\dim V = n > 0$ and choose $v \in V$, $v \neq 0$. Then:

$$\mathcal{L} = v, Tv, T^2 v, \ldots, T^n v$$

is linearly dependent because the length of $\mathcal{L}$ is $n + 1$. Thus there exists $a_0, \ldots, a_n \in \mathbb{C}$, not all zero, such that

$$0 = a_0 v + a_1 Tv + a_2 T^2 v + \cdots + a_n T^n v$$

Consider the polynomial $p \in \mathcal{P}(\mathbb{C})$ with coefficients given by $a_0, \ldots, a_n$. By the Fundamental Theorem of Algebra,

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m), \quad \forall\, z \in \mathbb{C},$$

where $m \leq n$, $c \in \mathbb{C}$, $c \neq 0$, and $\lambda_k \in \mathbb{C}$. Thus:

$$\begin{aligned} 0 &= a_0 v + a_1 Tv + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T - \lambda_1 I) \cdots (T - \lambda_m I) v \end{aligned}$$

Thus $(T - \lambda_k I)v = 0$ for at least one k, which means $T - \lambda_k I$ is not injective, which implies that λ_k is eigenvalue of T. $\square$

Example: Theorem 12 is not true for real vector spaces! Take for example the following operator $T \in \mathcal{L}(\mathbb{F}^2)$ defined as:

$$T(w, z) = (-z, w)$$

If $\mathbb{F} = \mathbb{R}$, then T is a counterclockwise rotation by 90 degrees. Since a 90 degree rotation of any nonzero $v \in \mathbb{R}^2$ will never equal a scalar multiple of itself, T has no eigenvalues!

On the other hand, if $\mathbb{F} = \mathbb{C}$, then by Theorem 12 T must have at least one eigenvalue. Indeed it has two, $\lambda = i$ and $\lambda = -i$ [see the book p. 135].

Recall we want a nice decomposition of V as $V = U_1 \oplus \cdots \oplus U_m$, where each U_k is an invariant subspace of T, so that to understand $T \in \mathcal{L}(V)$ we only need to understand $T|_{U_k}$. We will accomplish this by finding bases of V that yield matrices $\mathcal{M}(T)$ with lots of zeros.

As a first baby step, let V be a complex vector space. Then $T \in \mathcal{L}(V)$ must have at least one eigenvalue λ and a corresponding eigenvector v_*. Extend v_* to a basis of V:

$$\mathcal{B}_V = v_*, v_2, \ldots, v_n$$

Then:

$$\mathcal{M}(T; \mathcal{B}_V) = \begin{pmatrix} \lambda & & \\ 0 & & * \\ \vdots & & \\ 0 & & \end{pmatrix} \tag{9}$$

Furthermore, if we define $U_1 = \operatorname{span}(v_*)$ and $U_2 = \operatorname{span}(v_2, \ldots, v_n)$, then $V = U_1 \oplus U_2$. The subspace U_1 is a one dimensional invariant subspace of V under T, but U_2 is not necessarily. It is a start though! Now let's try to do better...

Definition 31. A matrix is upper triangular if all the entries below the diagonal equal 0:

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

There is a useful connection between upper triangular matrices and invariant subspaces:

Proposition 29. *Suppose $T \in \mathcal{L}(V)$ and $\mathcal{B}_V = v_1, \ldots, v_n$ is a basis for V. Then the following are equivalent:*

1. $\mathcal{M}(T; \mathcal{B}_V)$ *is upper triangular*
2. $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ *for each* $k = 1, \ldots, n$
3. $\operatorname{span}(v_1, \ldots, v_k)$ *is invariant under T for each* $k = 1, \ldots, n$

Proof. First we prove $1 \iff 2$. Let $A = \mathcal{M}(T; \mathcal{B}_V)$. Then by the definition of A we have:

$$Tv_k = \sum_{j=1}^{n} A_{j,k} v_j$$

But then

$$Tv_k \in \operatorname{span}(v_1, \ldots, v_k) \iff \underbrace{A_{j,k} = 0 \ \forall\, j > k}_{A \text{ is upper triangular}}$$

Clearly $3 \implies 2$

We finish the proof by showing $2 \implies 3$. Fix k. From 2 we have:

$$\begin{aligned} Tv_1 &\in \operatorname{span}(v_1) \subset \operatorname{span}(v_1, \ldots, v_k) \\ Tv_2 &\in \operatorname{span}(v_1, v_2) \subset \operatorname{span}(v_1, \ldots, v_k) \\ &\vdots \\ Tv_k &\in \operatorname{span}(v_1, \ldots, v_k) \end{aligned}$$

Thus if $v \in \operatorname{span}(v_1, \ldots, v_k)$, then $Tv \in \operatorname{span}(v_1, \ldots, v_k)$ as well. □

Now can improve upon our "baby step" (9) above by showing that given an eigenvector v_* with eigenvalue λ, we can extend it to a basis $\mathcal{B}_V$ such that $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular.

Theorem 13. *Suppose V is a finite dimensional complex vector space and $T \in \mathcal{L}(V)$. Then there exists a basis $\mathcal{B}_V$ such that $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular.*

End of Lecture 11