BEGINNING OF LECTURE 11

5.B Eigenvectors and Upper-Triangular Matrices

One of the main differences between operators and general linear maps is that we can take powers of operators! This will lead to many interesting results...

**Definition 29.** Let \( T \in \mathcal{L}(V) \) and let \( m \in \mathbb{Z}, m > 0 \).

- \( T^m = T \cdots T \) (composition \( m \) times)
- \( T^0 = I \)
- If \( T \) is invertible, then \( T^{-m} = (T^{-1})^m \)

**Definition 30.** Suppose \( T \in \mathcal{L}(V) \) and let \( p \in \mathcal{P}(\mathbb{F}) \) be given by:

\[
p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m
\]

Then \( p(T) \in \mathcal{L}(V) \) is defined as:

\[
p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m
\]

**Theorem 12.** Let \( V \neq \{0\} \) be a finite dimensional vector space over \( \mathbb{C} \). Then every \( T \in \mathcal{L}(V) \) has an eigenvalue.

**Proof.** Suppose \( \dim V = n > 0 \) and choose \( v \in V, v \neq 0 \). Then:

\[
\mathcal{L} = v,Tv,T^2v,\ldots,T^nv
\]

is linearly dependent because the length of \( \mathcal{L} \) is \( n + 1 \). Thus there exists \( a_0, \ldots, a_n \in \mathbb{C} \), not all zero, such that

\[
0 = a_0 v + a_1 T v + a_2 T^2 v + \cdots + a_n T^nv
\]

Consider the polynomial \( p \in \mathcal{P}(\mathbb{C}) \) with coefficients given by \( a_0, \ldots, a_n \). By the Fundamental Theorem of Algebra,

\[
p(z) = a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m), \quad \forall z \in \mathbb{C},
\]

where \( m \leq n, c \in \mathbb{C}, c \neq 0, \) and \( \lambda_k \in \mathbb{C} \). Thus:

\[
0 = a_0 v + a_1 T v + \cdots + a_n T^nv
\]

\[
= (a_0 I + a_1 T + \cdots + a_n T^m)v
\]

\[
= c(T - \lambda_1 I) \cdots (T - \lambda_m I)v
\]

Thus \( (T - \lambda_k I)v = 0 \) for at least one \( k \), which means \( T - \lambda_k I \) is not injective, which implies that \( \lambda_k \) is eigenvalue of \( T \). \( \square \)
Example: Theorem 12 is not true for real vector spaces! Take for example the following operator $T \in \mathcal{L}(\mathbb{F}^2)$ defined as:

$$T(w, z) = (-z, w)$$

If $\mathbb{F} = \mathbb{R}$, then $T$ is a counterclockwise rotation by 90 degrees. Since a 90 degree rotation of any nonzero $v \in \mathbb{R}^2$ will never equal a scalar multiple of itself, $T$ has no eigenvalues!

On the other hand, if $\mathbb{F} = \mathbb{C}$, then by Theorem 12 $T$ must have at least one eigenvalue. Indeed it has two, $\lambda = i$ and $\lambda = -i$ [see the book p. 135].

Recall we want a nice decomposition of $V$ as $V = U_1 \oplus \cdots \oplus U_m$, where each $U_k$ is an invariant subspace of $T$, so that to understand $T \in \mathcal{L}(V)$ we only need to understand $T|_{U_k}$. We will accomplish this by finding bases of $V$ that yield matrices $\mathcal{M}(T)$ with lots of zeros.

As a first baby step, let $V$ be a complex vector space. Then $T \in \mathcal{L}(V)$ must have at least one eigenvalue $\lambda$ and a corresponding eigenvector $v_*$. Extend $v_*$ to a basis of $V$:

$$\mathcal{B}_V = v_*, v_2, \ldots, v_n$$

Then:

$$\mathcal{M}(T; \mathcal{B}_V) = \begin{pmatrix} \lambda & & * \\ 0 & \ddots & \\ \vdots & \ddots & * \\ 0 & & \lambda_m \end{pmatrix} \quad \quad \quad (9)$$

Furthermore, if we define $U_1 = \text{span}(v_*)$ and $U_2 = \text{span}(v_2, \ldots, v_n)$, then $V = U_1 \oplus U_2$. The subspace $U_1$ is a one dimensional invariant subspace of $V$ under $T$, but $U_2$ is not necessarily. It is a start though! Now let’s try to do better...

**Definition 31.** A matrix is upper **triangular** if all the entries below the diagonal equal 0:

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & \cdots & \lambda_m \end{pmatrix}$$

There is a useful connection between upper triangular matrices and invariant subspaces:
Proposition 29. Suppose $T \in \mathcal{L}(V)$ and $\mathcal{B}_V = v_1, \ldots, v_n$ is a basis for $V$. Then the following are equivalent:

1. $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular

2. $Tv_k \in \text{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$

3. $\text{span}(v_1, \ldots, v_k)$ is invariant under $T$ for each $k = 1, \ldots, n$

Proof. First we prove $1 \iff 2$. Let $A = \mathcal{M}(T; \mathcal{B}_V)$. Then by the definition of $A$ we have:

$$Tv_k = \sum_{j=1}^{n} A_{j,k}v_j$$

But then

$$Tv_k \in \text{span}(v_1, \ldots, v_k) \iff A_{j,k} = 0 \quad \forall j > k$$

Clearly $3 \implies 2$

We finish the proof by showing $2 \implies 3$. Fix $k$. From 2 we have:

$$Tv_1 \in \text{span}(v_1) \subset \text{span}(v_1, \ldots, v_k)$$
$$Tv_2 \in \text{span}(v_1, v_2) \subset \text{span}(v_1, \ldots, v_k)$$
$$\vdots$$
$$Tv_k \in \text{span}(v_1, \ldots, v_k)$$

Thus if $v \in \text{span}(v_1, \ldots, v_k)$, then $Tv \in \text{span}(v_1, \ldots, v_k)$ as well. 

Now can improve upon our “baby step” (9) above by showing that given an eigenvector $v_*$ with eigenvalue $\lambda$, we can extend it to a basis $\mathcal{B}_V$ such that $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular.

Theorem 13. Suppose $V$ is a finite dimensional complex vector space and $T \in \mathcal{L}(V)$. Then there exists a basis $\mathcal{B}_V$ such that $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular.

End of Lecture 11