Beginning of Lecture 12

Warmup: Suppose $T \in \mathcal{L}(V)$ and $6I - 5T + T^2 = 0$. What are the possible eigenvalues of T?

Answer: $6I - 5T + T^2 = 0$ implies that $(T - 2I)(T - 3I) = 0$. Now let $v \neq 0$ be an eigenvector of T with eigenvalue λ. Then $0 = (T - 2I)(T - 3I)v = (\lambda - 2)(\lambda - 3)v$, which implies that $\lambda = 2$ or $\lambda = 3$.

Theorem 14. *Suppose V is a finite dimensional complex vector space and $T \in \mathcal{L}(V)$. Then there exists a basis $\mathcal{B}_V$ such that $\mathcal{M}(T; \mathcal{B}_V)$ is upper triangular.*

Proof. Induction on $\dim V$. Clearly the result is true when $\dim V = 1$.

Now suppose the result is true for all complex vector spaces with dimension $n - 1$ or less, and let V be a complex vector space with $\dim V = n$. We know that V has one eigenvalue λ. Define:

$$U = \operatorname{range}(T - \lambda I)$$

Since $T - \lambda I$ is not surjective, $\dim U < \dim V$. Furthermore, U is invariant under T; indeed, let $u \in U$:

$$Tu = \underbrace{(T - \lambda I)u}_{\in U} + \underbrace{\lambda u}_{\in U}$$

Thus $\widetilde{T} = T|_U \in \mathcal{L}(U)$, and we can apply the induction hypothesis to $\widetilde{T}$ and U. In particular, there exists a basis $\mathcal{B}_U = u_1, \ldots, u_m$ of U such that $\mathcal{M}(\widetilde{T}; \mathcal{B}_U)$ is upper triangular.

Extend $\mathcal{B}_U$ to a basis for V:

$$\mathcal{B}_V = u_1, \ldots, u_m, v_1, \ldots, v_\ell, \quad \ell + m = n$$

Since $\mathcal{M}(\widetilde{T}; \mathcal{B}_U)$ is upper triangular, by Proposition 29 we have:

$$Tu_k = \widetilde{T}u_k \in \operatorname{span}(u_1, \ldots, u_k) \text{ for all } k = 1, \ldots, m.$$

Furthermore,

$$Tv_j = \underbrace{(T - \lambda I)v_j}_{\in U} + \underbrace{\lambda v_j}_{\in \operatorname{span}(v_j)} \in \operatorname{span}(u_1, \ldots, u_m, v_j) \subset \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_j)$$

Thus T and $\mathcal{B}_V$ satisfy condition 2 of Proposition 29, and so $\mathcal{M}(T;\mathcal{B}_V)$ is upper triangular. □

Upper triangular matrices a very useful for determining if $T \in \mathcal{L}(V)$ is invertible...

Proposition 30. *Let $T \in \mathcal{L}(V)$ and let $\mathcal{B}$ be a basis for which $\mathcal{M}(T;\mathcal{B})$ is upper triangular. Then*

$$T \text{ is invertible} \iff \text{all diagonal entries of } \mathcal{M}(T;\mathcal{B}) \text{ are nonzero}$$

Proof. Let $\mathcal{B} = v_1, \ldots, v_n$ and let $A = \mathcal{M}(T;\mathcal{B})$. Easier to prove "not (a) $\iff$ not (b)".

First suppose T is not invertible; we want to show that some entry of $\mathcal{M}(T;\mathcal{B})$ is zero. T not invertible $\Rightarrow$ T not injective $\Rightarrow$ there exists $v \neq 0$ such that $Tv = 0$. Expand v in $\mathcal{B}$:

$$v = \sum_{j=1}^{n} c_j v_j$$

Let k be the index satisfying the following: $c_k \neq 0$ and $c_j = 0$ for all $j > k$ (note that possibly $k = n$). If $k = 1$, then $v = c_1 v_1 \Rightarrow Tv_1 = 0 \Rightarrow A_{1,1} = 0$. If $k > 1$ then:

$$\begin{aligned}
v &= \sum_{j=1}^{k} c_j v_j \\
Tv &= \sum_{j=1}^{k} c_j T v_j \\
0 &= \sum_{j=1}^{k-1} c_j T v_j + c_k T v_k \\
\Rightarrow Tv_k &= -\sum_{j=1}^{k-1} \left(\frac{c_j}{c_k}\right) T v_j \in \operatorname{span}(v_1, \ldots, v_{k-1}),
\end{aligned}$$

where in the last line we used Proposition 29. But also by Proposition 29,

$$\sum_{j=1}^{k-1} b_j v_j = Tv_k = \sum_{j=1}^{k} A_{j,k} v_j$$

and since $\mathcal{B}$ is a basis we must have $A_{k,k} = 0$.

Now suppose some entry on the diagonal of $\mathcal{M}(T;\mathcal{B})$ is zero. If $A_{1,1} = 0$ then $Tv_1 = 0$ and so T is not injective, and hence not invertible. If $A_{k,k} = 0$ for $k > 1$, then by Proposition 29 we have:

$$Tv_k = \sum_{j=1}^{k} A_{j,k} v_j = \sum_{j=1}^{k-1} A_{j,k} v_j \in \operatorname{span}(v_1, \ldots, v_{k-1}) \tag{10}$$

Consider now the linear map $\widetilde{T} = T|_{\operatorname{span}(v_1,\ldots v_k)}$. By (10),

$$\widetilde{T} \in \mathcal{L}(\operatorname{span}(v_1, \ldots, v_k), \operatorname{span}(v_1, \ldots, v_{k-1}))$$

Thus $\widetilde{T}$ cannot be injective since it maps a k-dimensional vector space to a $(k-1)$-dimensional vector space. In particular, there exists $v_* \in \operatorname{span}(v_1, \ldots, v_k)$ such that $\widetilde{T}v_* = 0$. But then $Tv_* = 0$, and so T is not injective, and hence not invertible. □

End of Lecture 12