Beginning of Lecture 13

Not only can upper triangular matrices tell us when $T \in \mathcal{L}(V)$ is invertible, they also tell us precisely what the eigenvalues of T are!

Proposition 31. *Let $T \in \mathcal{L}(V)$ and suppose $A = \mathcal{M}(T)$ is upper triangular. Then:*

$$\lambda \text{ is an eigenvalue of } T \iff \lambda = A_{k,k} \text{ for some } k$$

Proof. Let $A = \mathcal{M}(T)$ have diagonal entries given by $A_{k,k} = \lambda_k$:

$$A = \mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

Let $\lambda \in \mathbb{F}$. Then

$$\mathcal{M}(T - \lambda I) = \begin{pmatrix} \lambda_1 - \lambda & & * \\ & \ddots & \\ 0 & & \lambda_m - \lambda \end{pmatrix}$$

Thus by Proposition 30 $T - \lambda I$ is not invertible (and hence λ is an eigenvalue) if and only if $\lambda = \lambda_k$ for some k. □

5.C Eigenspaces and Diagonal Matrices

Definition 32. A <u>diagonal matrix</u> is a square matrix that is 0 everywhere except possibly the diagonal:

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

<u>Note:</u> If $\mathcal{M}(T; \mathcal{B})$ is upper triangular, then the diagonal entries are precisely the eigenvalues of T (since diagonal matrices are upper triangular).

Definition 33. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The <u>eigenspace</u> of T corresponding to λ is:

$$E(\lambda, T) = \operatorname{null}(T - \lambda I)$$

<u>Note:</u> $T|_{E(\lambda,T)} = \lambda I$ (so eigenspaces are invariant subspaces)

Proposition 32. *Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then:*

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T) \tag{11}$$

is a direct sum and furthermore

$$\dim E(\lambda_1, T) + \cdots + \dim(\lambda_m, T) \leq \dim V$$

Proof. Let $u_k \in E(\lambda_k, T)$ and suppose that

$$u_1 + \cdots + u_m = 0$$

Since eigenvectors corresponding to distinct eigenvalues are linearly independent, each $u_k = 0$ and so (11) is a direct sum.

Furthermore, by #16 of 2.C (HW1),

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) = \dim(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)) \leq \dim V$$

□

End of Lecture 13