Beginning of Lecture 14

Definition 34. An operator $T \in \mathcal{L}(V)$ is diagonalizable if there exists a basis $\mathcal{B}$ such that $\mathcal{M}(T;\mathcal{B})$ is diagonal.

Proposition 33. *Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Then: T is diagonalizable $\Leftrightarrow$ V has a basis of eigenvectors of T.*

Proof. An operator $T \in \mathcal{L}(V)$ has a diagonal matrix with respect to a basis $\mathcal{B} = v_1, \ldots, v_n$ if and only if $Tv_k = \lambda_k v_k$ for each k. □

Example: Not every operator is diagonalizable, even over complex vector spaces! Consider $T \in \mathcal{L}(\mathbb{C}^2)$ defined as:

$$T(w, z) = (z, 0)$$

Then $T^2 = 0$. Now let $v \neq 0$ be an eigenvector with eigenvalue λ. Then $0 = T^2 v = T(Tv) = \lambda T v = \lambda^2$. Thus $\lambda = 0$. Even though $\dim E(0, T^2) = 2$, we see that

$$E(0, T) = \{(w, 0) : w \in \mathbb{C}\}$$

and so $\dim(0, T) = 1$. Therefore V does not have a basis of eigenvectors of T, and so T is not diagonalizable. We will address examples like this much later with the notion of generalized eigenvectors...

On the other hand, if we have enough distinct eigenvalues, we know that T is diagonalizable:

Proposition 34. *If $T \in \mathcal{L}(V)$ has $\dim V < \infty$ distinct eigenvalues, then T is diagonalizable.*

Proof. Let $\dim V = n$ and suppose $T \in \mathcal{L}(V)$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $v_1, \ldots, v_n$. The eigenvectors are linearly independent because they correspond to distinct eigenvalues, and thus they form a basis for V. Thus T is diagonalizable. □

Note: The converse is not true! Take any diagonal matrix with non-unique entries on the diagonal.

Finally, our main result for this chapter. Namely, if T is diagonalizable, then we can achieve our stated goal of decomposing V as $V = U_1 \oplus \cdots \oplus U_n$, where each U_k is an invariant subspace of V under T and $\dim U_k = 1$.

Theorem 15. *Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote distinct eigenvalues of T. Then the following are equivalent:*

1. *T is diagonalizable*
2. *V has a basis consisting of eigenvectors of T*
3. *There exist one dimensional invariant subspaces $U_1, \ldots, U_n$ of V such that $V = U_1 \oplus \cdots \oplus U_n$*
4. *$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$*
5. *$\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$*

Proof. Many parts. The plan is:

$$1 \iff 2 \iff 3, \quad 2 \implies 4 \implies 5 \implies 2$$

- $1 \iff 2$: Simply Proposition 33.
- $2 \implies 3$: Let $\mathcal{B} = v_1, \ldots, v_n$ be basis of eigenvectors of V. Define $U_k = \text{span}(v_k)$. Then each U_k is a 1-dimensional invariant subspace of V under T, and since $\mathcal{B}$ is a basis it is clear $V = U_1 \oplus \cdots \oplus U_n$.
- $3 \implies 2$: For each k, let $v_k \in U_k$, $v_k \neq 0$. Since U_k is a 1-dimensional invariant subspace under T, each v_k is an eigenvector of T. Furthermore each $v \in V$ can be written uniquely as:

  $$v = u_1 + \cdots + u_n,$$

  where $u_k \in U_k$ and therefore $u_k = a_k v_k$ for some $a_k \in \mathbb{F}$. Thus $v_1, \ldots, v_n$ is a basis for V.
- $2 \implies 4$: Let $v_1, \ldots, v_n$ be a basis of eigenvectors for V, and subdivide the list according to the unique eigenvalues of T, so that:

  $$v_1^{(\ell)}, \ldots, v_{k_\ell}^{(\ell)} \text{ corresponds to } \lambda_\ell, \quad \text{for } \ell = 1, \ldots, m$$

  and $k_1 + k_2 + \cdots + k_m = n$. Then any $v \in V$ can be written as:

  $$v = \sum_{\ell=1}^{m} \underbrace{\sum_{j=1}^{k_\ell} a_{j,\ell}\, v_j^{(\ell)}}_{\in E(\lambda_\ell, T)} \in E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

- 4 $\Longrightarrow$ 5: This is simply 2.C #16, which you did for homework!
- 5 $\Longrightarrow$ 2: Choose a basis for each $E(\lambda_\ell, T)$, say $v_1^{(\ell)}, \ldots, v_{k_\ell}^{(\ell)}$, where $k_1 + \cdots + k_m = n$ by assumption. Let $\mathcal{L}$ be the list of all of these vectors concatenated together. To show $\mathcal{L}$ is linearly independent, suppose:

$$\sum_{\ell=1}^{m} \underbrace{\sum_{j=1}^{k_\ell} a_{j,\ell}\, v_j^{(\ell)}}_{u_\ell \in E(\lambda_\ell, T)} = 0$$

$$\sum_{\ell=1}^{m} u_\ell = 0$$

Each u_ℓ is eigenvector of T corresponding to a distinct eigenvalue λ_ℓ; thus $u_1, \ldots, u_m$ must be linearly independent and so $u_\ell = 0$ for all ℓ. But then $a_{j,\ell} = 0$ for all $j = 1, \ldots, k_\ell$ and for each ℓ, since $v_1^{(\ell)}, \ldots, v_{k_\ell}^{(\ell)}$ are linearly independent.

□

End of Lecture 14