Beginning of Lecture 15

6 Inner Product Spaces

We now introduce geometrical aspects such as length and angle into the setting of abstract vector spaces.

6.A Inner Products and Norms

We begin by looking at $\mathbb{R}^n$.

Definition 35. The norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is:

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition 36. For $x, y \in \mathbb{R}^n$, the dot product of x and y is:

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n.$$

Notice that $\|x\|^2 = x \cdot x$.

Example: In $\mathbb{R}^2$, $\|x\| = \sqrt{x_1^2 + x_2^2}$ which is just the length of x, and

$$x \cdot y = \|x\| \|y\| \cos\theta,$$

where θ is the angle between x and y.

Properties of the dot product:

- $x \cdot x \geq 0 \ \forall x \in \mathbb{R}^n$
- $x \cdot x = \|x\|^2 = 0 \Longleftrightarrow x = 0$
- $x \cdot y = y \cdot x$
- Fix $y \in \mathbb{R}^n$. Then $T_y(x) = x \cdot y$ is a linear map, i.e., $T_y \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Now we want to generalize the dot product to abstract vector spaces. First lets consider $\mathbb{C}^n$. Let $\lambda = a + ib \in \mathbb{C}$ be a complex scalar. Recall that:

- $|\lambda| = \sqrt{a^2 + b^2}$

- $|\lambda|^2 = \lambda\bar{\lambda}$

For $z \in \mathbb{C}^n$, the norm is defined as:

$$\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

Note that:

$$\|z\|^2 = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$$

If we want $z \cdot z = \|z\|^2$, then the previous line implies that we should define the dot product on $\mathbb{C}^n$ as:

$$w \cdot z = w_1\bar{z}_1 + \cdots + w_n\bar{z}_n$$

This leads us to the generalization of the dot product to abstract vector spaces:

Definition 37. An inner product on V is a function $\langle\cdot,\cdot\rangle : \mathbb{F}^2 \to \mathbb{F}$ that has the following properties:

1. Positive Definitness:
   $\langle v, v\rangle \geq 0 \ \forall\, v \in V$
   $\langle v, v\rangle = 0 \iff v = 0$

2. Linearity in the first argument:
   $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \ \forall\, u, v, w \in V$
   $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle \ \forall \lambda \in \mathbb{F}, \ \forall\, u, v \in V$

3. Conjugate Symmetry:
   $\langle u, v\rangle = \overline{\langle v, u\rangle} \ \forall\, u, v \in V$

Examples:

1. Euclidean inner product on $\mathbb{F}^n$. Let $w = (w_1, \ldots, w_n, z = (z_1, \ldots, z_n) \in \mathbb{F}^n$:

$$\langle w, z\rangle = w_1\bar{z}_1 + \cdots + w_n\bar{z}_n$$

2. Weighted Euclidean inner product on $\mathbb{F}^n$. Fix $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ with $c_k \geq 0$. Then for $w, z \in \mathbb{F}^n$,

$$\langle w, z\rangle_c = c_1 w_1\bar{z}_1 + \cdots + c_n w_n\bar{z}_n$$

3. Define $V = L^2(\mathbb{R})$ as:

$$L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty\}$$

One can verify this is a real vector space. Since it is a subset of the vector space of all functions mapping $\mathbb{R}$ to $\mathbb{R}$, we need to show (1) it contains an additive identity (zero), (2) it is closed under addition, and (3) it is closed under scalar multiplication. Indeed, $f \equiv 0 \in L^2(\mathbb{R})$, and furthermore if $f \in L^2(\mathbb{R})$ then $\lambda f \in L^2(\mathbb{R})$ for any $\lambda \in \mathbb{R}$ since

$$\int_{-\infty}^{\infty} |\lambda f(x)|^2 \, dx = \lambda^2 \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

The trickiest part is that it is closed under addition; i.e., if $f, g \in L^2(\mathbb{R})$, then $f + g \in L^2(\mathbb{R})$. First note:

$$\int_{-\infty}^{\infty} |f(x)+g(x)|^2 \, dx = \underbrace{\int_{-\infty}^{\infty} |f(x)|^2 \, dx}_{\text{I}} + \underbrace{\int_{-\infty}^{\infty} |g(x)|^2 \, dx}_{\text{II}} + \underbrace{2\int_{-\infty}^{\infty} f(x)g(x) \, dx}_{\text{III}}$$

Since $f, g \in L^2(\mathbb{R})$, we know that the first two terms are finite. That leaves the third term. That this is finite follows from what's known in Real Analysis as Hölder's Inequality. However, we can in fact prove it with more elementary tools. First let $a, b \in \mathbb{R}$ and note that:

$$(a - b)^2 \geq 0 \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

Now let $f(x) = a$ and $g(x) = b$. Then:

$$\int_{-\infty}^{\infty} f(x)g(x) \, dx \leq \int_{-\infty}^{\infty} \frac{|f(x)|^2}{2} + \frac{|g(x)|^2}{2} \, dx < \infty \tag{12}$$

Thus $L^2(\mathbb{R})$ is a vector space! We can add an inner product to it by defining the inner product as:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)g(x) \, dx$$

By what we just showed in (12), the inner product is well defined. Furthermore, it is easy to verify that all of the properties of an inner product hold, except for "definiteness" property: $\langle f, f\rangle = 0 \Rightarrow f = 0$. This is a bit technical but follows from Real Analysis. Now $L^2(\mathbb{R})$ is what we call an inner product space. Any inner product can always be used to define the norm of a vector. In this case, we get the L^2-norm:

$$\|f\|_2 = \sqrt{\langle f, f\rangle} = \left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)^{1/2}$$

In fact $L^2(\mathbb{R})$ is a special inner product space called a Hilbert space, but we leave that for more advanced math classes...

Definition 38. An <u>inner product space</u> is a vector space V along with an inner product on V.

Important Note: For the rest of chapter 6, we assume V is an inner product space.

Definition 39. For $v \in V$ an inner product space, the <u>norm</u> of v is:

$$\|v\| = \sqrt{\langle v, v, \rangle}$$

END OF LECTURE 15