BEGINNING OF LECTURE 16

Proposition 35. The following basic properties hold:

1. For each fixed \( u \in V \), the function \( T_u(v) = \langle v, u \rangle \) is linear, i.e., \( T_u \in \mathcal{L}(V, \mathbb{F}) \).

2. \( \langle 0, v \rangle = 0 \quad \forall v \in V \)

3. \( \langle v, 0 \rangle = 0 \quad \forall v \in V \)

4. \( \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \quad \forall u, v, w \in V \)

5. \( \langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle \quad \forall \lambda \in \mathbb{F} \) and \( u, v \in V \)

6. \( \|v\| = 0 \iff v = 0 \)

7. \( \|\lambda v\| = |\lambda|\|v\| \quad \forall \lambda \in \mathbb{F} \)

Proof. The proofs are all very simple and in the book.

Definition 40. \( u, v \in V \) are **orthogonal** if \( \langle u, v \rangle = 0 \).

In plane geometry, two vectors are orthogonal if they are perpendicular, see Figure 2.

![Figure 2: Orthogonal line segments](image)

It is easy to see the following two basic facts:

- 0 is orthogonal to every \( v \in V \)

- 0 is the only vector in \( V \) orthogonal to itself

Theorem 16 (Pythagorean Theorem). Suppose \( u \) and \( v \) are orthogonal vectors in \( V \). Then:

\[
\|u + v\|^2 = \|u\|^2 + \|v\|^2
\]
Proof.

\[ \|u + v\|^2 = \langle u + v, u + v \rangle \]
\[ = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \]
\[ = \|u\|^2 + \|v\|^2 \]

Now consider the following problem: Suppose \( u, v \in V \) with \( v \neq 0 \). We want to write \( u \) as:

\[ u = cv + w, \quad \langle v, w \rangle = 0 \]

From the book, we have the picture in Figure 3. The question is, what are \( c \) and \( w \)?

First write \( u \) as:

\[ u = cv + (u - cv) \]

We need to choose \( c \) such that:

\[ 0 = \langle u - cv, v \rangle = \langle u, v \rangle - c\|v\|^2 \Rightarrow c = \frac{\langle u, v \rangle}{\|v\|^2} \]

We summarize this in the following proposition:

**Proposition 36.** Suppose \( u, v \in V \) with \( v \neq 0 \). Set:

\[ c = \frac{\langle u, v \rangle}{\|v\|^2} \quad \text{and} \quad w = u - \frac{\langle u, v \rangle}{\|v\|^2}v. \]
Then:
\[ \langle w, v \rangle = 0 \quad \text{and} \quad u = cv + w \]

**Theorem 17 (Cauchy-Schwarz Inequality).** Suppose \( u, v \in V \). Then:
\[ |\langle u, v \rangle| \leq \|u\|\|v\| \]

Furthermore,
\[ |\langle u, v \rangle| = \|u\|\|v\| \iff u = cv \]

**Proof.** If \( v = 0 \) then both sides are zero. Thus assume \( v \neq 0 \), and apply the orthogonal decomposition to \( u \):
\[ u = \frac{\langle u, v \rangle}{\|v\|^2} + w, \quad \langle v, w \rangle = 0 \]

By the Pythagorean Theorem:
\[
\|u\|^2 = \left| \frac{\langle u, v \rangle}{\|v\|^2} \right|^2 + \|w\|^2 \\
= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \\
\geq \frac{|\langle u, v \rangle|^2}{\|v\|^2} 
\]

Now multiply both sides by \( \|v\|^2 \).

For the second part, we see from the above proof that equality holds if and only if \( w = 0 \). But then:
\[ w = u - \frac{\langle u, v \rangle}{\|v\|^2} v = 0 \iff u = \frac{\langle u, v \rangle}{\|v\|^2} v \]

The Cauchy-Schwarz Inequality is one of the most important, and most used, inequalities in all of mathematics! Let’s now use it to prove the triangle inequality for general inner product spaces; Figure 6 gives the plane geometry intuition.
**Theorem 18** (Triangle Inequality). Suppose $u, v \in V$. Then:

$$
\|u + v\| \leq \|u\| + \|v\|,
$$

with equality if and only if $u = cv$ for $c \geq 0$.

The next result is the Parallelogram Equality, which also has a geometric interpretation in $\mathbb{R}^2$; see Figure 7.

**Proposition 37.** Suppose $u, v \in V$. Then:

$$
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)
$$

**End of Lecture 16**