Beginning of Lecture 16

Proposition 35. *The following basic properties hold:*

1. *For each fixed $u \in V$, the function $T_u(v) = \langle v, u \rangle$ is linear, i.e., $T_u \in \mathcal{L}(V, \mathbb{F})$.*
2. $\langle 0, v \rangle = 0 \;\; \forall\, v \in V$
3. $\langle v, 0 \rangle = 0 \;\; \forall\, v \in V$
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle \;\; \forall\, u, v, w \in V$
5. $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle \;\; \forall\, \lambda \in \mathbb{F}$ *and* $u, v \in V$
6. $\|v\| = 0 \iff v = 0$
7. $\|\lambda v\| = |\lambda| \|v\| \;\; \forall\, \lambda \in \mathbb{F}$

Proof. The proofs are all very simple and in the book. □

Definition 40. $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$.

In plane geometry, two vectors are orthogonal if they are perpendicular, see Figure 2.

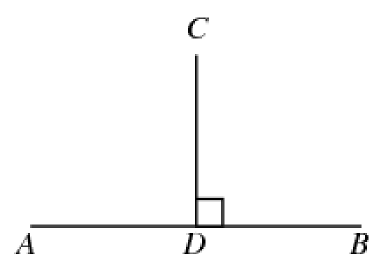

Figure 2: Orthogonal line segments

It is easy to see the following two basic facts:

- 0 is orthogonal to every $v \in V$
- 0 is the only vector in V orthogonal to itself

Theorem 16 (Pythagorean Theorem)**.** *Suppose u and v are orthogonal vectors in V. Then:*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

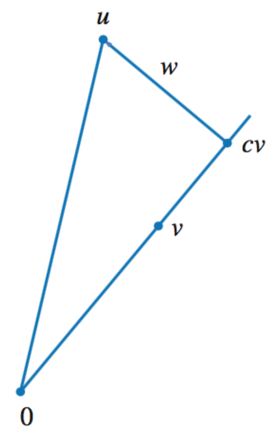

Figure 3: Orthogonal decomposition of u

Proof.

$$\begin{aligned} \|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

□

Now consider the following problem: Suppose $u, v \in V$ with $v \neq 0$. We want to write u as:

$$u = cv + w, \quad \langle v, w\rangle = 0$$

From the book, we have the picture in Figure 3. The question is, what are c and w?

First write u as:

$$u = cv + (u - cv)$$

We need to choose c such that:

$$0 = \langle u - cv, v\rangle = \langle u, v\rangle - c\|v\|^2 \Rightarrow c = \langle u, v\rangle / \|v\|^2$$

We summarize this in the following proposition:

Proposition 36. *Suppose $u, v \in V$ with $v \neq 0$. Set:*

$$c = \frac{\langle u, v\rangle}{\|v\|^2} \quad \textit{and} \quad w = u - \frac{\langle u, v\rangle}{\|v\|^2} v.$$

Then:

$$\langle w, v \rangle = 0 \quad \textit{and} \quad u = cv + w$$

Theorem 17 (Cauchy-Schwarz Inequality)**.** *Suppose $u, v \in V$. Then:*

$$|\langle u, v \rangle| \leq \|u\|\|v\|$$

Furthermore,

$$|\langle u, v \rangle| = \|u\|\|v\| \iff u = cv$$

Proof. If $v = 0$ then both sides are zero. Thus assume $v \neq 0$, and apply the orthogonal decomposition to u:

$$u = \frac{\langle u, v \rangle}{\|v\|^2} + w, \quad \langle v, w \rangle = 0$$

By the Pythagorean Theorem:

$$\begin{aligned}
\|u\|^2 &= \left\| \frac{\langle u, v \rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\
&= \frac{|\langle u, v \rangle|^2}{\|v\|^2} + \|w\|^2 \\
&\geq \frac{|\langle u, v \rangle|^2}{\|v\|^2}
\end{aligned}$$

Now multiply both sides by $\|v\|^2$.

For the second part, we see from the above proof that equality holds if and only if $w = 0$. But then:

$$w = u - \frac{\langle u, v \rangle}{\|v\|^2} v = 0 \iff u = \frac{\langle u, v \rangle}{\|v\|^2} v$$

□

The Cauchy-Schwarz Inequality is one of the most important, and most used, inequalities in all of mathematics! Lets now use it to prove the triangle inequality for general inner product spaces; Figure 6 gives the plane geometry intuition.

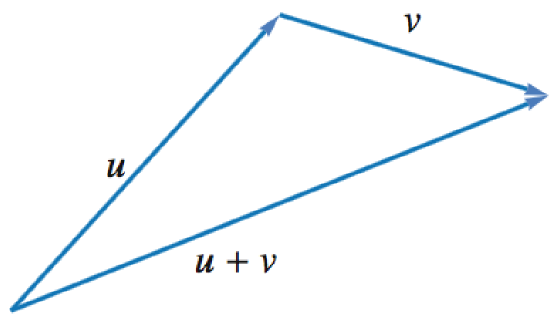

Figure 4: The triangle inequality for $\mathbb{R}^2$

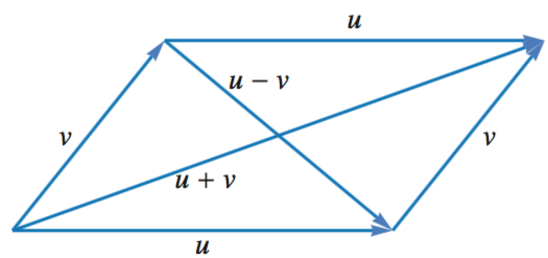

Figure 5: Parallelogram equality in $\mathbb{R}^2$

Theorem 18 (Triangle Inequality). *Suppose $u, v \in V$. Then:*

$$\|u + v\| \leq \|u\| + \|v\|,$$

with equality if and only if $u = cv$ for $c \geq 0$.

The next result is the Parallelogram Equality, which also has a geometric interpretation in $\mathbb{R}^2$; see Figure 7.

Proposition 37. *Suppose $u, v \in V$. Then:*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

End of Lecture 16