**Beginning of Lecture 19**

**Theorem 19** (Triangle Inequality). Suppose \( u, v \in V \). Then:

\[
\|u + v\| \leq \|u\| + \|v\|,
\]

with equality if and only if \( u = cv \) for \( c \geq 0 \).

**Proof.** For the first part:

\[
\|u + v\|^2 = \langle u + v, u + v \rangle \\
= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \langle v, u \rangle \\
= \langle u, u \rangle + \langle v, v \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} \\
= \|u\|^2 + \|v\|^2 + 2\text{Re}\langle u, v \rangle \\
\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v \rangle| \\
\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \quad \text{[Cauchy-Schwarz]} \\
= (\|u\| + \|v\|)^2
\]

The proof above shows that equality holds if and only if:

1. \( \text{Re}\langle u, v \rangle = |\langle u, v \rangle| \), and
2. \( |\langle u, v \rangle| = \|u\|\|v\| \)

From the Cauchy-Schwarz inequality, we know #2 holds if and only if \( u = cv \) for some \( c \in \mathbb{F} \). For #1, consider an arbitrary \( \lambda = a + ib \in \mathbb{C} \), where \( a, b \in \mathbb{R} \). Then \( \text{Re}\lambda = a \) and \( |\lambda| = \sqrt{a^2 + b^2} \), so \( \text{Re}\lambda = |\lambda| \) if and only if \( \lambda = a \geq 0 \). Thus #1 holds if and only if \( \langle u, v \rangle \geq 0 \), which combined with \( u = cv \), implies that equality holds if and only if \( c \geq 0 \).

The next result is the Parallelogram Equality, which also has a geometric interpretation in \( \mathbb{R}^2 \); see Figure 7.

**Proposition 38.** Suppose \( u, v \in V \). Then:

\[
\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)
\]

**Proof.** Simply compute:

\[
\|u + v\|^2 + \|u - v\|^2 = \langle u + v, u + v \rangle + \langle u - v, u - v \rangle \\
= \|u\|^2 + \|v\|^2 + \langle u, v \rangle + \langle v, u \rangle + \|u\|^2 + \|v\|^2 - \langle u, v \rangle + \langle v, u \rangle \\
= 2(\|u\|^2 + \|v\|^2)
\]

\(\square\)
6. B Orthonormal Bases

**Definition 41.** A list of vectors $e_1, \ldots, e_m \in V$ is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is an inner product. The norm 1 and orthogonal conditions are satisfied.

where

$$\delta : \mathbb{Z} \to \mathbb{C}, \quad \delta(0) = 1 \text{ and } \delta(n) = 0, \quad \forall n \neq 0.$$

Examples:

1. The standard basis in $\mathbb{F}^n$

2. Recalls the vector space $V = \{ f : \mathbb{Z}_N \to \mathbb{C} \}$, where $\mathbb{Z}_N = \{0, \ldots, N - 1\}$, and the Fourier basis:

$$e_k : \mathbb{Z}_N \to \mathbb{C}, \quad e_k(n) = \frac{1}{\sqrt{N}} e^{2\pi i kn/N}.$$

Define an inner product on this vector space:

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f(n) \overline{g(n)}$$
Now $V$ is an inner product space and $e_0, \ldots, e_{N-1}$ is an orthonormal list. We can verify this:

$$\langle e_j, e_k \rangle = \sum_{n=0}^{N-1} e_j(n) e_k(n)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i jn/N} e^{-2\pi i k n/N}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (j-k)n/N}$$

$$= \left\{ \begin{array}{ll}
\frac{1}{N} \sum_{n=0}^{N-1} 1 = \frac{1}{N} \cdot N = 1 & \text{if } j = k \\
\frac{1}{N} \cdot \frac{1 - (e^{2\pi i (j-k)/N})^N}{1 - e^{2\pi i (j-k)/N}} = \frac{1}{N} \cdot \frac{1 - e^{2\pi i (j-k)}}{1 - e^{2\pi i (j-k)/N}} = \frac{1}{N} \cdot \frac{1 - 1}{1 - e^{2\pi i (j-k)/N}} = 0 & \text{if } j \neq k
\end{array} \right.$$  

Since $e_0, \ldots, e_{N-1}$ is also a basis, we call it an orthonormal basis.

**Definition 42.** An **orthonormal basis** of $V$ is an orthonormal list of vectors in $V$ that is also a basis of $V$.

**End of Lecture 19**