Beginning of Lecture 19

Theorem 19 (Triangle Inequality)**.** *Suppose $u, v \in V$. Then:*

$$\|u+v\| \leq \|u\| + \|v\|,$$

with equality if and only if $u = cv$ for $c \geq 0$.

Proof. For the first part:

$$\begin{aligned}
\|u+v\|^2 &= \langle u+v, u+v\rangle \\
&= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \langle v,u\rangle \\
&= \langle u,u\rangle + \langle v,v\rangle + \langle u,v\rangle + \overline{\langle u,v\rangle} \\
&= \|u\|^2 + \|v\|^2 + 2\text{Re}\langle u,v\rangle \\
&\leq \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle| \\
&\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| \qquad \text{[Cauchy-Schwarz]} \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

The proof above shows that equality holds if and only if:

1. $\text{Re}\langle u,v\rangle = |\langle u,v\rangle|$, and
2. $|\langle u,v\rangle| = \|u\|\|v\|$

From the Cauchy-Schwartz inequality, we know #2 holds if and only if $u = cv$ for some $c \in \mathbb{F}$. For #1, consider an arbitrary $\lambda = a + ib \in \mathbb{C}$, where $a, b \in \mathbb{R}$. Then $\text{Re}\lambda = a$ and $|\lambda| = \sqrt{a^2+b^2}$, so $\text{Re}\lambda = |\lambda|$ if and only if $\lambda = a \geq 0$. Thus #1 holds if and only if $\langle u,v\rangle \geq 0$, which combined with $u = cv$, implies that equality holds if and only if $c \geq 0$. □

The next result is the Parallelogram Equality, which also has a geometric interpretation in $\mathbb{R}^2$; see Figure 7.

Proposition 38. *Suppose $u, v \in V$. Then:*

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

Proof. Simply compute:

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \|u\|^2 + \|v\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|u\|^2 + \|v\|^2 - \langle u,v\rangle + \langle v,u\rangle \\
&= 2(\|u\|^2 + \|v\|^2)
\end{aligned}$$

□

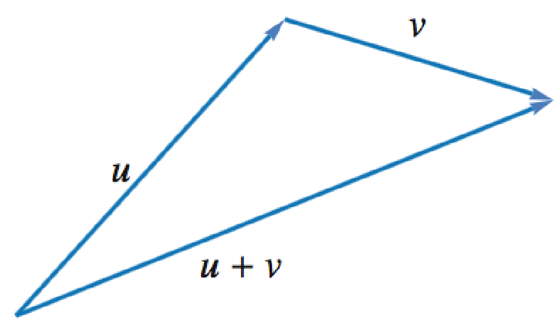

Figure 6: The triangle inequality for $\mathbb{R}^2$

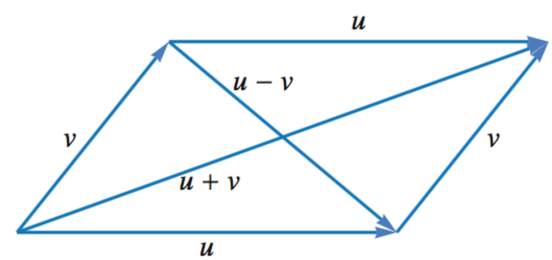

Figure 7: Parallelogram equality in $\mathbb{R}^2$

6.B Orthonormal Bases

Definition 41. A list of vectors $e_1, \ldots, e_m \in V$ is orthonormal if

$$\langle e_j, e_k \rangle = \left\{ \begin{array}{lll} 1 & \text{if } j = k & \text{[norm 1]} \\ 0 & \text{if } j \neq k & \text{[orthogonal]} \end{array} \right\} = \delta(j-k),$$

where

$$\delta : \mathbb{Z} \to \mathbb{C}, \quad \delta(0) = 1 \text{ and } \delta(n) = 0, \ \forall\, n \neq 0.$$

Examples:

1. The standard basis in $\mathbb{F}^n$
2. Recalls the vector space $V = \{f : \mathbb{Z}_N \to \mathbb{C}\}$, where $\mathbb{Z}_N = \{0, \ldots, N-1\}$, and the Fourier basis:

$$e_k : \mathbb{Z}_N \to \mathbb{C}, \quad e_k(n) = \frac{1}{\sqrt{N}} e^{2\pi i k n / N}.$$

   Define an inner product on this vector space:

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f(n) \overline{g(n)}$$

Now V is an inner product space and $e_0, \ldots, e_{N-1}$ is an orthonormal list. We can verify this:

$$\begin{aligned}
\langle e_j, e_k \rangle &= \sum_{n=0}^{N-1} e_j(n)\overline{e_k(n)} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi ijn/N} e^{-2\pi ikn/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i(j-k)n/N} \\
&= \begin{cases} \frac{1}{N}\sum_{n=0}^{N-1} 1 = \frac{1}{N}\cdot N = 1 & \text{if } j = k \\ \frac{1}{N}\cdot \frac{1-(e^{2\pi i(j-k)/N})^N}{1-e^{2\pi i(j-k)/N}} = \frac{1}{N}\cdot\frac{1-e^{2\pi i(j-k)}}{1-e^{2\pi i(j-k)/N}} = \frac{1}{N}\cdot\frac{1-1}{1-e^{2\pi i(j-k)/N}} = 0 & \text{if } j \neq k \end{cases}
\end{aligned}$$

Since $e_0, \ldots, e_{N-1}$ is also a basis, we call it an orthonormal basis.

Definition 42. An orthonormal basis of V is an orthonormal list of vectors in V that is also a basis of V.

End of Lecture 19