Beginning of Lecture 20

Definition 43. An <u>orthonormal basis</u> of V is an orthonormal list of vectors in V that is also a basis of V.

Orthonormal lists and bases are very convenient! For example:

Proposition 39. *If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then:*

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2, \quad \forall\, a_1, \ldots, a_m \in \mathbb{F}.$$

Proof. Expand the left hand side:

$$\begin{aligned}
\left\| \sum_{k=1}^m a_k e_k \right\|^2 &= \sum_{j=1}^m \sum_{k=1}^m \langle a_j e_j, a_k e_k \rangle \\
&= \sum_{j=1}^m \sum_{k=1}^m a_j \overline{a}_k \langle e_j, e_k \rangle \\
&= \sum_{j=1}^m \sum_{k=1}^m a_j \overline{a}_k \delta(j-k) \\
&= \sum_{k=1}^m |a_k|^2
\end{aligned}$$

□

Corollary 6 (Important!). *Every orthonormal list of vectors is linearly independent.*

Proof. Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V and $a_1, \ldots, a_m \in \mathbb{F}$ are such that:

$$\sum_{k=1}^m a_k e_k = 0.$$

Then by the previous proposition, $|a_1|^2 + \cdots + |a_m|^2 = 0$, which means $a_k = 0$ for all k since $|a_k|^2 \geq 0$. Thus $e_1, \ldots, e_m$ are linearly independent. □

Proposition 40. *If $\dim V = n$ and $e_1, \ldots, e_n \in V$ is an orthonormal list of vectors, then $e_1, \ldots, e_m$ is an orthonormal basis.*

Proof. By the previous corollary such a list must be linearly independent, and since $n = \dim V$ it then must be a basis. □

In general, given a basis $v_1, \ldots, v_n$ of V and a vector $v \in V$, we know there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n$$

However, computing $a_1, \ldots, a_n$ can be difficult. If we use an orthonormal basis though, the calculation becomes very easy!

Theorem 20. *Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then:*

$$v = \sum_{k=1}^{n} \underbrace{\langle v, e_k \rangle}_{a_k} e_k$$

and

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2$$

Proof. Because $e_1, \ldots, e_n$ is a basis of V, there exists $a_1, \ldots, a_n \in \mathbb{F}$ such that

$$v = \sum_{k=1}^{n} a_k e_k$$

Now compute the inner product of both sides of the previous equation with e_j:

$$\langle v, e_j \rangle = \langle \sum_{k=1}^{n} a_k e_k, e_j \rangle = \sum_{k=1}^{n} a_k \langle e_k, e_j \rangle = \sum_{k=1}^{n} a_k \delta(j-k) = a_j$$

The second equation on $\|v\|^2$ now follows immediately from Proposition 39. □

Since orthonormal bases are so useful, how do we go about finding them? The next algorithm shows how to turn any linearly independent list into an orthonormal list with the same span.

Theorem 21 (Gram-Schmidt)**.** *Suppose $v_1, \ldots, v_m \in V$ are linearly independent. Define:*

$$e_1 = \frac{v_1}{\|v_1\|},$$

and then for $k = 2, \ldots, m$, define e_k inductively by:

$$e_k = \frac{v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j}{\left\| v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j \right\|} \tag{13}$$

Then $e_1, \ldots, e_m \in V$ *is an orthonormal list of vectors such that:*

$$\mathrm{span}(v_1, \ldots, v_k) = \mathrm{span}(e_1, \ldots, e_k), \quad \forall\, k = 1, \ldots, m.$$

Remark: Step two of the Gram-Schmidt algorithm is:

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} = \frac{1}{\|v_2 - \langle v_2, e_1 \rangle e_1\|} \cdot \left(v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1 \right)$$

which looks very similar to our orthogonal decomposition theorem from 6.A. The only difference is that e_2 is normalized to have norm one. The idea of Gram-Schmidt is to iterate on this decomposition. Now for the proof:

Proof. Proof by induction on k. For $k = 1$, clearly $\mathrm{span}(v_1) = \mathrm{span}(e_1)$, and so our base case holds.

Now suppose that for $1 < k < m$ we have

$$\mathrm{span}(v_1, \ldots, v_{k-1}) = \mathrm{span}(e_1, \ldots, e_{k-1}),$$

and let us consider $e_1, \ldots, e_k$. First note that $v_k \notin \mathrm{span}(v_1, \ldots, v_{k-1})$ (because they are linearly independent) and thus $v_k \notin \mathrm{span}(e_1, \ldots, e_{k-1})$. Thus denominator of e_k in (13) is not zero and so it is well defined. Clearly it has norm one, i.e., $\|e_k\| = 1$.

Now let $1 \leq j < k$. Then:

$$\begin{aligned}
\langle e_k, e_j \rangle &= \left\langle \frac{v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j}{\left\| v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j \right\|}, e_k \right\rangle \\
&= \frac{\langle v_k, e_j \rangle - \langle v_k, e_j \rangle}{\left\| v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j \right\|} \\
&= 0
\end{aligned}$$

Thus $e_1, \ldots, e_k$ is an orthonormal list.

By the definition of e_k, we have:

$$v_k = \left\| v_k - \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j \right\| e_k + \sum_{j=1}^{k-1} \langle v_k, e_j \rangle e_j$$

and so $v_k \in \text{span}(e_1, \dots, e_k)$. Thus by the inductive hypothesis,

$$\text{span}(v_1, \dots, v_k) \subset \text{span}(e_1, \dots, e_k).$$

But both lists $v_1, \dots, v_k$ and $e_1, \dots, e_k$ are linearly independent, and thus both subspaces have dimension k. Therefore they must be equal. □

END OF LECTURE 20