Beginning of Lecture 21

Example: Let's use Gram-Schmidt find an orthonormal basis of $\mathcal{P}_2([-1,1];\mathbb{R})$ with the inner product:

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Let's start with the standard basis $1, x, x^2$ which is linearly independent but not orthonormal. We start by computing:

$$\|1\|^2 = \int_{-1}^{1} 1^2\,dx = 2$$

Thus:

$$e_1 = 1/\|1\| = 1/\sqrt{2}$$

Now we need to compute e_2. So we compute:

$$x - \langle x, e_1\rangle e_1 = x - \left(\int_{-1}^{1} x\frac{1}{\sqrt{2}}\,dx\right)\frac{1}{\sqrt{2}} = x$$

and also:

$$\|x - \langle x, e_1\rangle e_1\|^2 = \|x\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}.$$

Thus:

$$e_2 = \frac{x - \langle x, e_1\rangle e_1}{\|x - \langle x, e_1\rangle e_1\|} = \sqrt{\frac{3}{2}}x$$

Now we need to compute e_3. We have:

$$\begin{aligned} x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2 &= x^2 - \left(\int_{-1}^{1} x^2\frac{1}{\sqrt{2}}\,dx\right)\frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} x^2\sqrt{\frac{3}{2}}x\,dx\right)\sqrt{\frac{3}{2}}x \\ &= x^2 - \frac{1}{3} \end{aligned}$$

and also

$$\begin{aligned}\|x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2\|^2 &= \left\| x^2 - \frac{1}{3} \right\|^2 \\ &= \int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 \, dx \\ &= \int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9} \right) \, dx = \frac{8}{45}.\end{aligned}$$

Hence:

$$e_3 = \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right).$$

Thus

$$\mathcal{B} = \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - 1/3)$$

is an orthonormal basis for $\mathcal{P}_2([-1,1];\mathbb{R})$.

The Gram-Schmidt algorithm can be used to prove several useful facts, which we do now.

Proposition 41. *Every finite dimensional inner product space has an orthonormal basis.*

Proof. Choose any basis of V and apply the Gram-Schmidt algorithm to it to get an orthonormal basis. □

Just as we can extend any linearly independent list to a basis, we can also extend any orthonormal list to an orthonormal basis.

Proposition 42. *If V is a finite dimensional inner product space, then every list of orthonormal vectors in V can be extended to an orthonormal basis of V.*

Proof. Let $e_1, \ldots, e_m \in V$ be an orthonormal list. Since they are linearly independent, we can extend them to a basis:

$$e_1, \ldots, e_m, v_1, \ldots, v_n$$

Now apply the Gram-Schmidt algorithm to this basis. Since $e_1, \ldots, e_m$ are orthonormal, as you can verify the Gram-Schmidt algorithm will leave them unchanged. Thus we get an orthonormal basis of the form:

$$e_1, \ldots, e_m, f_1, \ldots, f_n$$

□

Now we return to upper-triangular matrices. Recall that we previously showed that if V is a finite dimensional complex vector space, then for each $T \in \mathcal{L}(V)$ there is a basis $\mathcal{B}$ such that $\mathcal{M}(T; \mathcal{B})$ is upper triangular. When V is an inner product space, we would like to take $\mathcal{B}$ to be an orthonormal basis.

Proposition 43. *Suppose $T \in \mathcal{L}(V)$. If $\mathcal{M}(T; \mathcal{B})$ is upper triangular for some basis $\mathcal{B}$, then there exists an orthonormal basis $\mathcal{B}'$ such that $\mathcal{M}(T; \mathcal{B}')$ is upper triangular.*

Proof. Suppose $\mathcal{M}(T; \mathcal{B})$ is upper triangular and $\mathcal{B} = v_1, \ldots, v_n$. Then $U_k = \text{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

Apply the Gram-Schmidt algorithm to $\mathcal{B}$, producing an orthonormal basis $\mathcal{B}' = e_1, \ldots, e_n$. We claim $\mathcal{B}'$ is the desired basis. Indeed,

$$\text{span}(e_1, \ldots, e_k) = \text{span}(v_1, \ldots, v_k) = U_k, \quad \forall\, k = 1, \ldots, n.$$

Therefore $\text{span}(e_1, \ldots, e_k)$ is invariant under T for each $k = 1, \ldots, n$. Thus $\mathcal{M}(T; \mathcal{B}')$ is upper triangular. □

Remark: The above proposition holds for *any* inner product space and operator T for which there exists some basis $\mathcal{B}$ such that $\mathcal{M}(T; \mathcal{B})$ is upper triangular. In particular, V can be a real vector space, if such a $\mathcal{B}$ exists. Of course when V is a complex vector space, we can guarantee the result...

Theorem 22 (Schur's Theorem)**.** *If V is a finite dimensional complex inner product space and $T \in \mathcal{L}(V)$, then there exists an orthonormal basis $\mathcal{B}'$ such that $\mathcal{M}(T; \mathcal{B}')$ is upper triangular.*

Proof. Since V is a finite dimensional complex vector space, there exists a basis $\mathcal{B}$ such that $\mathcal{M}(T; \mathcal{B})$ is upper triangular. Now apply the previous proposition. □

End of Lecture 21