Beginning of Lecture 22

Definition 44. A function φ is a linear functional on V if $\varphi \in \mathcal{L}(V, \mathbb{F})$.

Examples:

- *Fix* an arbitrary $u \in V$. Then:

$$\begin{aligned} \varphi : V &\to \mathbb{F} \\ v &\mapsto \varphi(v) = \langle v, u \rangle \end{aligned}$$

  is a linear functional on V.

- *Fix* an arbitrary continuous function $f \in C([-1,1];\mathbb{R})$. Then:

$$\begin{aligned} \varphi : \mathcal{P}_2([-1,1];\mathbb{R}) &\to \mathbb{R} \\ p &\mapsto \varphi(p) = \int_{-1}^{1} p(x) f(x)\, dx \end{aligned}$$

  is a linear functional on $\mathcal{P}([-1,1];\mathbb{R})$.

  Remark: It is tempting to write $\varphi(p) = \langle p, f \rangle$, but we may not have $f \in \mathcal{P}_2([-1,1];\mathbb{R})$. For example, $f(x) = \cos(x)$ or $f(x) = e^x$, and so $\langle p, f \rangle$ does not necessarily make sense. Thus the next result is quite remarkable...

Theorem 23 (Riesz Representation Theorem)**.** *Suppose V is finite-dimensional and $\varphi \in \mathcal{L}(V, \mathbb{F})$. Then there is a unique vector $u \in V$ such that*

$$\varphi(v) = \langle v, u \rangle, \quad \forall v \in V$$

Proof. First we show that there exists a $u \in V$ such that $\varphi(v) = \langle v, u \rangle$, then we show that u is unique. Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Then:

$$\begin{aligned} \varphi(v) &= \varphi\left(\sum_{k=1}^{n} \langle v, e_k \rangle e_k\right) \\ &= \sum_{k=1}^{n} \langle v, e_k \rangle \varphi(e_k) \\ &= \langle v, \sum_{k=1}^{n} \overline{\varphi(e_k)} e_k \rangle \end{aligned}$$

Thus setting:

$$u = \sum_{k=1}^{n} \overline{\varphi(e_k)} e_k$$

we have $\varphi(v) = \langle v, u \rangle$ for all $v \in V$.

Now we prove that u is unique. Suppose $u_1, u_2 \in V$ such that

$$\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle, \quad \forall\, v \in V$$

Then:

$$0 = \langle v, u_1 \rangle - \langle v, u_2 \rangle = \langle v, u_1 - u_2 \rangle, \quad \forall\, v \in V$$

Taking $v = u_1 - u_2$ implies $\|u_1 - u_2\|^2 = 0$ which implies that $u_1 - u_2 = 0$ and so $u_1 = u_2$. Therefore u is unique. $\square$

Remark (con't): Returning to the example above, even if $f \in C([-1,1];\mathbb{R})$ and $f \notin \mathcal{P}_2([-1,1];\mathbb{R})$, there still exists a unique $q \in \mathcal{P}_2([-1,1];\mathbb{R})$ such that:

$$\varphi(p) = \int_{-1}^{1} p(x) f(x)\, dx = \int_{-1}^{1} p(x) q(x)\, dx = \langle p, q \rangle, \quad \forall\, p \in \mathcal{P}_2([-1,1];\mathbb{R}).$$

Furthermore, we can compute what q is by selecting an orthonormal basis e_1, e_2, e_3 for $\mathcal{P}_2([-1,1];\mathbb{R})$ (like the one we computed earlier) and using the formula:

$$q = \overline{\varphi(e_1)} e_1 + \overline{\varphi(e_2)} e_2 + \overline{\varphi(e_3)} e_3$$

This works in general.

End of Lecture 22