BEGINNING OF LECTURE 23

6.C Orthogonal Complements and Minimization Problems

Definition 45. If $U \subset V$, then the orthogonal complement of $U$ is:

$$U^\perp = \{ v \in V : \langle v, u \rangle = 0, \ \forall u \in U \}$$

Geometrical Examples:

- If $U$ is a line in $V = \mathbb{R}^2$, then $U^\perp$ is the line orthogonal to $U$ that passes through the origin.
- If $U$ is a line in $V = \mathbb{R}^3$, then $U^\perp$ is the plane orthogonal to $U$ that contains the origin.
- If $U$ is a plane in $V = \mathbb{R}^3$, then $U^\perp$ is the line orthogonal to $U$ that passes through the origin.

Proposition 44. The following are basic properties of the orthogonal complement:

1. If $U \subset V$, then $U^\perp$ is a subspace of $V$
2. $\{0\}^\perp = V$
3. $V^\perp = \{0\}$
4. If $U \subset V$, then $U \cap U^\perp \subset \{0\}$. If $U$ is a subspace of $V$, then $U \cap U^\perp = \{0\}$.
5. If $U \subset V$ and $W \subset V$ ad $U \subset W$, then $W^\perp \subset U^\perp$

Proof. We go through the list:

1. We need to show $U^\perp$ contains 0, is closed under addition and scalar multiplication.
   - Clearly $\langle 0, u \rangle = 0$ $\forall u \in U$, thus $0 \in U^\perp$.
   - Now suppose $v, w \in U^\perp$. If $u \in U$, then:
     $$\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle = 0 + 0 = 0 \implies v + w \in U^\perp$$
Suppose \( \lambda \in \mathbb{F} \) and \( v \in U^\perp \). If \( u \in U \), then
\[
\langle \lambda v, u \rangle = \lambda \langle v, u \rangle = \lambda \cdot 0 = 0 \implies \lambda v \in U^\perp
\]
\section{2.}
\( \langle v, 0 \rangle = 0 \ \forall v \in V \implies v \in \{0\}^\perp \), so \( \{0\}^\perp = V \)
\section{3.}
Suppose \( v \in V^\perp \). Then \( \langle v, v \rangle = 0 \implies v = 0 \). Thus \( V^\perp = \{0\} \).
\section{4.}
Suppose \( U \subset V \) and \( v \in U \cap U^\perp \). Then we must have \( \langle v, v \rangle = 0 \implies v = 0 \), and so \( U \cap U^\perp \subset \{0\} \). If \( U \) is a subspace of \( V \), then \( 0 \in U \) and by above \( 0 \in U^\perp \), so \( U \cap U^\perp = \{0\} \).
\section{5.}
This is clear.

Recall early on we proved that if \( U \) is a subspace of \( V \), then there exists a second subspace \( W \) of \( V \) such that \( V = U \oplus W \). We now show that we can take \( W = U^\perp \).

\textbf{Proposition 45.} \textit{If \( U \) is a finite dimensional subspace of \( V \), then}
\[
V = U \oplus U^\perp
\]
\textit{Proof.} From the previous proposition we know that \( U \cap U^\perp = \{0\} \), so we just need to show that \( U + U^\perp = V \). Let \( v \in V \) and let \( e_1, \ldots, e_m \) be an ONB of \( U \). Clearly then:
\[
v = \sum_{k=1}^{m} \langle v, e_k \rangle e_k + v - \sum_{k=1}^{m} \langle v, e_k \rangle e_k
\]
We want to show \( w \in U^\perp \). But this is clear since:
\[
\forall k = 1, \ldots, m, \quad \langle w, e_k \rangle = \langle v, e_k \rangle - \langle v, e_k \rangle = 0 \implies w \in U^\perp
\]
\textbf{Corollary 7.} \textit{If \( V \) is finite dimensional and \( U \) is a subspace of \( V \), then:}
\[
\dim U^\perp = \dim V - \dim U
\]
\textbf{Proposition 46.} \textit{If \( U \) is a finite dimensional subspace of \( V \), then}
\[
U = (U^\perp)^\perp
\]
Proof. We prove this in two parts:

- First we show that $U \subset (U^\perp)^\perp$. Suppose $u \in U$. Then by definition of $U^\perp$,
  \[ \langle v, u \rangle = 0 = \langle u, v \rangle, \quad \forall v \in U^\perp \]
  But the above also implies that $u \in (U^\perp)^\perp$ since
  \[ (U^\perp)^\perp = \{ w \in V : \langle w, v \rangle = 0, \quad \forall v \in U^\perp \} \]

- Now we show that $(U^\perp)^\perp \subset U$. Suppose that $v \in (U^\perp)^\perp$. $v \in V$ so we can write it as:
  \[ v = u + w, \quad u \in U, \ w \in U^\perp \implies v - u = w \in U^\perp \]
  But by the above, we also have $u \in U \subset (U^\perp)^\perp$ and so:
  \[ v - u \in (U^\perp)^\perp \Rightarrow v - u \in U^\perp \cap (U^\perp)^\perp \Rightarrow v - u = 0 \Rightarrow v = u \Rightarrow v \in U \]

End of Lecture 23