Beginning of Lecture 23

6.C Orthogonal Complements and Minimization Problems

Definition 45. If $U \subset V$, then the orthogonal complement of U is:

$$U^\perp = \{v \in V : \langle v, u\rangle = 0, \ \forall\, u \in U\}$$

Geometrical Examples:

- If U is a line in $V = \mathbb{R}^2$, then $U^\perp$ is the line orthogonal U that passes through the origin.
- If U is a line in $V = \mathbb{R}^3$, then $U^\perp$ is the plane orthogonal to U that contains the origin.
- If U is a plane in $V = \mathbb{R}^3$, then $U^\perp$ is the line orthogonal to U that passes through the origin.

Proposition 44. *The following are basic properties of the orthogonal complement:*

1. *If $U \subset V$, then $U^\perp$ is a subspace of V*
2. $\{0\}^\perp = V$
3. $V^\perp = \{0\}$
4. *If $U \subset V$, then $U \cap U^\perp \subset \{0\}$. If U is a subspace of V, then $U \cap U^\perp = \{0\}$.*
5. *If $U \subset V$ and $W \subset V$ ad $U \subset W$, then $W^\perp \subset U^\perp$*

Proof. We go through the list:

1. We need to show $U^\perp$ contains 0, is closed under addition and scalar multiplication.
    - Clearly $\langle 0, u\rangle = 0 \ \forall\, u \in U$, thus $0 \in U^\perp$.
    - Now suppose $v, w \in U^\perp$. If $u \in U$, then:

$$\langle v + w, u\rangle = \langle v, u\rangle + \langle w, u\rangle = 0 + 0 = 0 \Longrightarrow v + w \in U^\perp$$

- Suppose $\lambda \in \mathbb{F}$ and $v \in U^\perp$. If $u \in U$, then

$$\langle \lambda v, u\rangle = \lambda\langle v, u\rangle = \lambda \cdot 0 = 0 \Longrightarrow \lambda v \in U^\perp$$

2. $\langle v, 0\rangle = 0 \ \forall\, v \in V \Longrightarrow v \in \{0\}^\perp$, so $\{0\}^\perp = V$
3. Suppose $v \in V^\perp$. Then $\langle v, v\rangle = 0 \Longrightarrow v = 0$. Thus $V^\perp = \{0\}$.
4. Suppose $U \subset V$ and $v \in U \cap U^\perp$. Then we must have $\langle v, v\rangle = 0 \Longrightarrow v = 0$, and so $U \cap U^\perp \subset \{0\}$. If U is a subspace of V, then $0 \in U$ and by above $0 \in U^\perp$, so $U \cap U^\perp = \{0\}$.
5. This is clear.

□

Recall early on we proved that if U is a subspace of V, then there exists a second subspace W of V such that $V = U \oplus W$. We now show that we can take $W = U^\perp$.

Proposition 45. *If U is a finite dimensional subspace of V, then*

$$V = U \oplus U^\perp$$

Proof. From the previous proposition we know that $U \cap U^\perp = \{0\}$, so we just need to show that $U + U^\perp = V$. Let $v \in V$ and let $e_1, \ldots, e_m$ be an ONB of U. Clearly then:

$$v = \underbrace{\sum_{k=1}^{m}\langle v, e_k\rangle e_k}_{u \in U} + \underbrace{v - \sum_{k=1}^{m}\langle v, e_k\rangle e_k}_{w}$$

We want to show $w \in U^\perp$. But this is clear since:

$$\forall\, k = 1, \ldots, m, \quad \langle w, e_k\rangle = \langle v, e_k\rangle - \langle v, e_k\rangle = 0 \Longrightarrow w \in U^\perp$$

□

Corollary 7. *If V is finite dimensional and U is a subspace of V, then:*

$$\dim U^\perp = \dim V - \dim U$$

Proposition 46. *If U is a finite dimensional subspace of V, then*

$$U = (U^\perp)^\perp$$

Proof. We prove this in two parts:

- First we show that $U \subset (U^\perp)^\perp$. Suppose $u \in U$. Then by definition of $U^\perp$,
$$\langle v, u\rangle = 0 = \langle u, v\rangle, \quad \forall\, v \in U^\perp$$
But the above also implies that $u \in (U^\perp)^\perp$ since
$$(U^\perp)^\perp = \{w \in V : \langle w, v\rangle = 0,\ \forall\, v \in U^\perp\}$$

- Now we show that $(U^\perp)^\perp \subset U$. Suppose that $v \in (U^\perp)^\perp$. $v \in V$ so we can write it as:
$$v = u + w, \quad u \in U,\, w \in U^\perp \Longrightarrow v - u = w \in U^\perp$$
But by the above, we also have $u \in U \subset (U^\perp)^\perp$ and so:
$$v - u \in (U^\perp)^\perp \Rightarrow v - u \in U^\perp \cap (U^\perp)^\perp \Rightarrow v - u = 0 \Rightarrow v = u \Rightarrow v \in U$$

□

End of Lecture 23