Beginning of Lecture 24

Now we use the fact that $V = U \oplus U^{\perp}$ to define the orthogonal projection of V onto U.

Definition 46. Suppose U is a finite dimensional subspace of V.
The orthogonal projection of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as:

$$P_U v = u, \quad \text{where } v = u + w, \;\; u \in U, \, w \in U^{\perp}$$

Remark: Since the decomposition $v = u + w \in U \oplus U^{\perp}$ is unique, the orthogonal projection P_U is well defined.

Example: Recall from earlier we have: If $u, v \in V$ and $u \neq 0$, then

$$v = cu + w, \quad \langle u, w \rangle = 0, \;\; c = \frac{\langle v, u \rangle}{\|u\|^2}$$

Thus if $U = \operatorname{span}(u)$, then

$$P_U v = cu = \frac{\langle v, u \rangle}{\|u\|^2} u$$

More generally, if U is an arbitrary finite dimensional subspace of V and $e_1, \ldots, e_m$ is an ONB for U, then:

$$P_U v = \sum_{k=1}^{m} \langle v, e_k \rangle e_k \tag{14}$$

This is just one of many properties of P_U:

Proposition 47. *If U is a finite dimensional subspace of V and $v \in V$, then:*

1. $P_U \in \mathcal{L}(V)$
2. $P_U u = u \;\; \forall\, u \in U$
3. $P_U w = 0 \;\; \forall\, w \in U^{\perp}$
4. $\operatorname{range} P_U = U$
5. $\operatorname{null} P_U = U^{\perp}$

6. $v - P_U v \in U^\perp$

7. $P_U^2 = P_U$

8. $\|P_U v\| \leq \|v\|$

Proof. We prove each part:

1. This follows from (14) and the linearity of the inner product in the first argument.

2. If $u \in U$, then $u = u + 0 \in U \oplus U^\perp$, and thus $P_U u = u$.

3. If $w \in U^\perp$, then $w = 0 + w \in U \oplus U^\perp$, and thus $P_U w = 0$.

4. This is clear

5. Part 3 implies that $U^\perp \subset \operatorname{null} P_U$. Now suppose that $v \in \operatorname{null} P_U$, i.e., $P_U v = 0$. Then if $v = u + w \in U \oplus U^\perp$, we must have $P_U v = u = 0$, which implies that $v = 0 + w = w \in U^\perp$ and so $\operatorname{null} P_U \subset U^\perp$.

6. If $v = u + w \in U \oplus U^\perp$, then:

$$v - P_U v = (u + w) - u = w \in U^\perp$$

7. If $v = u + w \in U \oplus U^\perp$ then:

$$(P_U^2)v = P_U(P_U v) = P_U u = u = P_U v$$

8. If $v = u + w \in U \oplus U^\perp$ then:

$$\|P_U v\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2$$

□

We now turn to a very important minimization problem: Given a subspace U of V and a point $v \in V$, find a point $u_0 \in U$ such that $\|v - u_0\|$ is as small as possible. In other words, find $u_0 \in U$ such that:

$$\|v - u_0\| = \min_{u \in U} \|v - u\| \iff \|v - u_0\| \leq \|v - u\|, \ \forall\, u \in U$$

In fact the orthogonal projection gives the solution!

Theorem 24. *Suppose U is a finite dimensional subspace of V, $v \in V$, and $u \in U$. Then:*

$$\|v - P_U v\| \leq \|v - u\|.$$

Furthermore,

$$\|v - P_U v\| = \|v - u\| \Longleftrightarrow u = P_U v$$

Proof. We have:

$$\begin{aligned} \|v - P_U v\|^2 &\leq \|\underbrace{v - P_U v}_{\in U^\perp}\|^2 + \|\underbrace{P_U v - u}_{\in U}\|^2 \\ &= \|(v - P_U v) + (P_U v - u)\|^2 \\ &= \|v - u\|^2 \end{aligned}$$

The inequality is an equality if and only if:

$$\begin{aligned} \|v - P_U v\| = \|v - u\| &\Leftrightarrow \|v - P_U v\|^2 = \|v - P_U v\|^2 + \|P_U v - u\|^2 \\ &\Leftrightarrow \|P_U v - u\|^2 = 0 \\ &\Leftrightarrow P_U v = u \end{aligned}$$

□

Please read the very interesting Example 6.58 in the book.

End of Lecture 24