BEGINNING OF LECTURE 25

7 Operators on Inner Product Spaces

We now explore the structure of operators on inner product spaces, which we have been building towards for quite a while now. This will lead to some of the most important results in all of Linear Algebra. In particular, we will completely characterize those operators that are diagonalizable, giving us a complete solution to the questions we asked at the beginning of Chapter 5.

Notation: V and W are inner product spaces over the same field $\mathbb{F}$. Sometimes we will write $\langle \cdot, \cdot \rangle_V$ for the inner product on V and $\langle \cdot, \cdot \rangle_W$ for the inner product on W.

7.A Self-Adjoint and Normal Operators

Definition 47. Suppose $T \in \mathcal{L}(V, W)$. The adjoint of T is the function $T^* : W \to V$ such that:

$$\langle Tv, w \rangle_W = \langle v, T^* w \rangle_V, \quad \forall\, v \in V, \ \forall\, w \in W$$

Warmup: Show the T^* is well-defined.

Solution: Fix $T \in \mathcal{L}(V, W)$ and $w \in W$ and consider the linear functional $\varphi \in \mathcal{L}(V, \mathbb{F})$ defined as:

$$\varphi(v) = \langle Tv, w \rangle$$

By the Riesz Representation Theorem, there exists a unique vector $u \in V$ such that:

$$\langle Tv, w \rangle = \varphi(v) = \langle v, u \rangle$$

Define $T^* w := u$.

Example: Consider the vector space:

$$V = \{ f \in C^\infty([0,1]; \mathbb{R}) : f^{(k)}(0) = f^{(k)}(1) \ \forall\, k = 0, 1, 2, \ldots \}$$

Define the inner product on V as:

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx, \quad f, g \in V$$

Let $T \in \mathcal{L}(V)$ be the differentiation operator, i.e., $T = D$ where

$$Df = f'$$

Let's compute the adjoint of D using integration by parts:

$$\begin{aligned}
\langle Df, g\rangle &= \int_0^1 Df(x)g(x)\,dx \\
&= \int_0^1 f'(x)g(x)\,dx \\
&= f(x)g(x)|_{x=0}^{x=1} - \int_0^1 f(x)g'(x)\,dx \\
&= 0 - \int_0^1 f(x)g'(x)\,dx \\
&= \int_0^1 f(x)\cdot(-Dg(x))\,dx \\
&= \langle f, -Dg\rangle
\end{aligned}$$

Thus, $D^* = -D$.

We have used this technique before, but it will be especially useful when dealing with adjoints and so we write it down here:

Lemma 2. *Let $u, w \in V$. If:*

$$\langle v, u\rangle = \langle v, w\rangle, \quad \forall\, v \in V,$$

then $u = w$.

Proof. Indeed,

$$\langle v, u\rangle = \langle v, w\rangle,\ \ \forall\, v \in V \Longrightarrow \langle v, u - w\rangle = 0,\ \ \forall\, v \in V$$

Since it is true for all $v \in V$, we can take $v = u - w$ and we then have:

$$\langle u - w, u - w\rangle = 0 \Longrightarrow \|u - w\|^2 = 0 \Longrightarrow u - w = 0 \Longrightarrow u = w$$

□

Proposition 48. *If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$*

Proof. Need to show additivity and homogeneity of T^*. Let $w_1, w_2 \in W$ and $v \in V$:

$$\begin{aligned}\langle v, T^*(w_1 + w_2)\rangle_V &= \langle Tv, w_1 + w_2\rangle_W \\ &= \langle Tv, w_1\rangle_W + \langle Tv, w_2\rangle_W \\ &= \langle v, T^*w_1\rangle_V + \langle v, T^*w_1\rangle_V \\ &= \langle v, T^*w_1 + T^*w_2\rangle_V\end{aligned}$$

Therefore $T^*(w_1 + w_2) = T^*w_1 + T^*w_2$.

Now let $w \in W$, $\lambda \in \mathbb{F}$, and $v \in V$:

$$\begin{aligned}\langle v, T^*(\lambda w)\rangle_V &= \langle Tv, \lambda w\rangle_W \\ &= \bar{\lambda}\langle Tv, w\rangle_W \\ &= \bar{\lambda}\langle v, T^*w\rangle_V \\ &= \langle v, \lambda T^*w\rangle_V\end{aligned}$$

and so $T^*(\lambda w) = \lambda T^* w$ □

Proposition 49. *The following properties of the adjoint hold:*

1. $(S + T)^* = S^* + T^*, \ \forall\, S, T \in \mathcal{L}(V, W)$
2. $(\lambda T)^* = \bar{\lambda} T^*, \ \forall\, \lambda \in \mathbb{F}, \ \forall\, T \in \mathcal{L}(V, W)$
3. $(T^*)^* = T, \ \forall\, T \in \mathcal{L}(V, W)$
4. $I^* = I$, *where* $I \in \mathcal{L}(V)$ *is the identity operator, i.e.,* $Iv = v \ \forall v \in V$
5. $(ST)^* = T^*S^*, \ \forall\, T \in \mathcal{L}(V, W), \ \forall\, S \in \mathcal{L}(W, U)$

Proof. The proofs of #1 and #2 are very similar to the proof that T^* is linear. The proof of #4 is also quite easy.

To prove #3, let $v \in V$ and $w \in W$,

$$\langle w, (T^*)^*v\rangle = \langle T^*w, v\rangle = \overline{\langle v, T^*w\rangle} = \overline{\langle Tv, w\rangle} = \langle w, Tv\rangle$$

To prove #5, let $v \in V$ and $u \in U$,

$$\langle v, (ST)^*u\rangle = \langle STv, u\rangle = \langle Tv, S^*u\rangle = \langle v, T^*S^*u\rangle$$

□

End of Lecture 25