Math 414: Linear Algebra II, Fall 2015
Midterm 1

October 12, 2015

Name: Solutions

This is a **closed book single author exam**. Use of books, notes, or other aids is *not* permissible, nor is collaboration with any of your fellow students.

You must **prove, justify, or explain** all of your assertions.

This midterm is out of 100 **points**.

[4 pts] Please write your **name** above, and at the **top** of each subsequent page.

QUESTION 1:

True or False (explain each of your answers):

(a) [6 pts] If the vectors $v_1, \ldots, v_m$ span the vector space V, then every vector in V can be written as a linear combination of the vectors $v_1, \ldots, v_m$ in only one way.

Solution:
FALSE
Spanning does not imply linear independence.

(b) [6 pts] There exist vectors v_1, v_2, v_3 that are linearly dependent, but such that $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, and $w_3 = v_3 + v_1$ are linearly independent.

Solution:
FALSE
v_1, v_2, v_3 linearly independent implies $\dim \operatorname{span}(v_1, v_2, v_3) < 3$.
$w_1, w_2, w_3 \in \operatorname{span}(v_1, v_2, v_3)$ implies $\operatorname{span}(w_1, w_2, w_3) \subset \operatorname{span}(v_1, v_2, v_3)$ and so $\dim \operatorname{span}(w_1, w_2, w_3) \leq \dim \operatorname{span}(v_1, v_2, v_3) < 3$.
Therefore w_1, w_2, w_3 cannot be linearly independent.

(c) [6 pts] If an operator has one eigenvector, it has infinitely many eigenvectors.

Solution:
TRUE
Let v be an eigenvector with eigenvalue λ. Then cv is an eigenvector with eigenvalue λ for all $c \in \mathbb{F}$.

(d) [6 pts] The sum of two eigenvectors of an operator is always an eigenvector.

Solution:
FALSE
Let v_1 be an eigenvector with eigenvalue λ_1, and let v_2 be an eigenvector with eigenvalue λ_2, such that $\lambda_1 \neq \lambda_2$. Then $v_1 + v_2$ is not an eigenvector.

QUESTION 2:

A matrix $A \in \mathbb{F}^{n,n}$ is *symmetric* if $A_{j,k} = A_{k,j}$ for all $j, k = 1, \ldots, n$. Let $\mathrm{Sym}(n)$ denote the set of all symmetric matrices, i.e.,

$$\mathrm{Sym}(n) = \{A \in \mathbb{F}^{n,n} : A_{j,k} = A_{k,j} \ \forall\, j, k = 1, \ldots, n\}.$$

(a) [6 pts] Prove that $\mathrm{Sym}(n)$ is a subspace of $\mathbb{F}^{n,n}$.

Solution:
Since $\mathrm{Sym}(n) \subset \mathbb{F}^{n,n}$, we just need to show three things: (1) $0 \in \mathrm{Sym}(n)$, (2) $\mathrm{Sym}(n)$ is closed under addition, (3) $\mathrm{Sym}(n)$ is closed under scalar multiplication.

- Clearly $0 \in \mathrm{Sym}(n)$.
- Let $A, B \in \mathrm{Sym}(n)$. Then:

$$(A + B)_{j,k} = A_{j,k} + B_{j,k} = A_{k,j} + B_{k,j} = (A + B)_{k,j}.$$

- Let $A \in \mathrm{Sym}(n)$ and $\lambda \in \mathbb{F}$. Then:

$$(\lambda A)_{j,k} = \lambda A_{j,k} = \lambda A_{k,j} = (\lambda A)_{k,j}.$$

(b) [6 pts] Write down a basis for $\mathrm{Sym}(3)$.

Solution:
First note that:

$$\mathrm{Sym}(3) = \left\{ \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{F} \right\}$$

Therefore a basis for $\mathrm{Sym}(3)$ is:

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) [6 pts] What is the dimension of $\mathrm{Sym}(3)$? Explain your answer.

Solution:
$\dim \mathrm{Sym}(3) = 6$ since $\#\mathcal{B} = 6$.

(d) [6 pts] Find a subspace U of $\mathbb{F}^{2,2}$ such that $\mathbb{F}^{2,2} = \mathrm{Sym}(2) \oplus U$. For whatever U you write down, make sure your prove that $\mathbb{F}^{2,2} = \mathrm{Sym}(2) \oplus U$.

Solution:
First note that:

$$\mathrm{Sym}(2) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}.$$

Now define U as:

$$U = \left\{ \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} : d \in \mathbb{F} \right\}.$$

Clearly $\mathrm{Sym}(2) \cap U = \{0\}$.
Now let $A \in \mathbb{F}^{2,2}$ with:

$$A = \begin{pmatrix} a & e \\ f & c \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} a & e \\ f & c \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ b & c \end{pmatrix}}_{\in \mathrm{Sym}(2)} + \underbrace{\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}}_{\in U},$$

if and only if for every $e, f \in \mathbb{F}$ there exists $b, d \in \mathbb{F}$ such that

$$\begin{aligned} b + d &= e, \\ b - d &= f. \end{aligned}$$

But this is just two equations and two unknowns, which we know is always solvable, so such a b, d exist for every e, f. Therefore $\mathbb{F}^{2,2} = \mathrm{Sym}(2) + U$ and so $\mathbb{F}^{2,2} = \mathrm{Sym}(2) \oplus U$.

QUESTION 3:

The set $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a 2-tuple $(x, y) \in \mathbb{R}^2$.

(a) [8 pts] Let $\alpha = a+ib \in \mathbb{C}$. Treating $\mathbb{C}$ as a *complex* vector space, show that the function

$$T : \mathbb{C} \to \mathbb{C}$$
$$T(z) = \alpha z$$

is a linear operator on $\mathbb{C}$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

Solution:
Need to show additivity and homogeneity to prove T is linear:

- $T(z + w) = \alpha(z + w) = \alpha z + \alpha w = T(z) + T(w)$.
- $T(\lambda z) = \alpha(\lambda z) = \lambda(\alpha z) = \lambda T(z)$.

The matrix of T is simply $\mathcal{M}(T) = (\alpha)$.

(b) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as the *real* vector space $\mathbb{R}^2$, show that the function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x + iy) = \alpha(x + iy)$$

is a linear operator on $\mathbb{R}^2$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

Solution:
First note that:

$$\alpha z = (a + ib)(x + iy) = ax - by + i(bx + ay).$$

Therefore, treating $\mathbb{C}$ as $\mathbb{R}^2$,

$$T(x, y) = (ax - by, bx + ay).$$

Now we show T is linear:

- For additivity:

$$\begin{aligned} T(x + x', y + y') &= (a(x + x') - b(y + y'), b(x + x') + a(y + y')), \\ &= (ax + ax' - by - by', bx + bx' + ay + ay'), \\ &= (ax - by, bx + ay) + (ax' - by', bx' + ay'), \\ &= T(x, y) + T(x', y'). \end{aligned}$$

- For homogeneity:

$$T(\lambda x, \lambda y) = (a(\lambda x) - b(\lambda y), b(\lambda x) + a(\lambda y)) = \lambda(ax - by, bx + ay) = \lambda T(x, y).$$

The matrix of T is:

$$\mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(c) [8 pts] Define

$$T(x + iy) = 2x - y + i(x - 3y).$$

Show that this function is not a linear operator on the complex vector space $\mathbb{C}$, but it is a linear operator if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$.

Solution:
Showing $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear is essentially the same proof as the proof of linearity for part (b).

$T : \mathbb{C} \to \mathbb{C}$ is a function between one dimensional vector spaces. We know from the homework that T is linear if and only if T is scalar multiplication, i.e., $T(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. But from part (b) we know that:

$$\alpha z = (a + ib)(x + iy) = ax - by + i(bx + ay),$$

and clearly T does not have this form. Therefore T cannot be linear.

QUESTION 4:

Let V be a finite dimensional complex vector space. An operator $T \in \mathcal{L}(V)$ is called *nilpotent* if $T^k = 0$ for some $k \in \mathbb{Z}$, $k \geq 1$, i.e., $T^k v = 0$ for all $v \in V$.

(a) [12 pts] What are the eigenvalues of a nilpotent operator? Make sure to prove your assertion.

Solution:
Let $\lambda \in \mathbb{C}$ be an eigenvalue with eigenvector $v \neq 0$. Then:

$$Tv = \lambda v \Rightarrow 0 = T^k v = \lambda^k v \Rightarrow \lambda^k = 0 \Rightarrow \lambda = 0.$$

Therefore the only eigenvalue of T is 0.

(b) [12 pts] For a nilpotent operator T, let $k_0 \in \mathbb{Z}$, $k_0 \geq 1$, be the *smallest* positive integer such that $T^{k_0} = 0$. So in particular:

$$\begin{aligned} &T^k \neq 0, \text{ for all } 1 \leq k < k_0, \\ &T^{k_0} = 0. \end{aligned}$$

For which k_0 is T diagonalizable and for which k_0 is T not diagonalizable? Make sure to prove your assertion.

Solution:
If $k_0 = 1$ then $T = 0$ and $\mathcal{M}(T) = 0$, which is a diagonal matrix. So for $k_0 = 1$, T is diagonalizable.

Now suppose $k_0 > 1$. Then $T \neq 0$. Now suppose T is diagonalizable. We know that the diagonal entries of $\mathcal{M}(T)$ correspond to the eigenvalues of T. But by part (a) the only eigenvalue of T is zero. Therefore $\mathcal{M}(T) = 0$. But this implies that $T = 0$, which is contradiction. Therefore T is not diagonalizable.