NAME: Solutions

This is a closed book single author exam. Use of books, notes, or other aids is not permissible, nor is collaboration with any of your fellow students.

You must prove, justify, or explain all of your assertions.

This midterm is out of 100 points.

[4 pts] Please write your name above, and at the top of each subsequent page.
QUESTION 1:

True or False (explain each of your answers):

(a) [6 pts] If the vectors $v_1, \ldots, v_m$ span the vector space $V$, then every vector in $V$ can be written as a linear combination of the vectors $v_1, \ldots, v_m$ in only one way.

Solution:
FALSE
Spanning does not imply linear independence.

(b) [6 pts] There exist vectors $v_1, v_2, v_3$ that are linearly dependent, but such that $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, and $w_3 = v_3 + v_1$ are linearly independent.

Solution:
FALSE
$v_1, v_2, v_3$ linearly independent implies $\dim \text{span}(v_1, v_2, v_3) < 3$.
$w_1, w_2, w_3 \in \text{span}(v_1, v_2, v_3)$ implies $\text{span}(w_1, w_2, w_3) \subset \text{span}(v_1, v_2, v_3)$ and so $\dim \text{span}(w_1, w_2, w_3) \leq \dim \text{span}(v_1, v_2, v_3) < 3$.
Therefore $w_1, w_2, w_3$ cannot be linearly independent.

(c) [6 pts] If an operator has one eigenvector, it has infinitely many eigenvectors.

Solution:
TRUE
Let $v$ be an eigenvector with eigenvalue $\lambda$. Then $cv$ is an eigenvector with eigenvalue $\lambda$ for all $c \in \mathbb{F}$.

(d) [6 pts] The sum of two eigenvectors of an operator is always an eigenvector.

Solution:
FALSE
Let $v_1$ be an eigenvector with eigenvalue $\lambda_1$, and let $v_2$ be an eigenvector with eigenvalue $\lambda_2$, such that $\lambda_1 \neq \lambda_2$. Then $v_1 + v_2$ is not an eigenvector.
**Question 2:**

A matrix \( A \in \mathbb{F}^{n,n} \) is **symmetric** if \( A_{j,k} = A_{k,j} \) for all \( j, k = 1, \ldots, n \). Let \( \text{Sym}(n) \) denote the set of all symmetric matrices, i.e.,

\[
\text{Sym}(n) = \{ A \in \mathbb{F}^{n,n} : A_{j,k} = A_{k,j} \ \forall \ j, k = 1, \ldots, n \}.
\]

(a) [6 pts] Prove that \( \text{Sym}(n) \) is a subspace of \( \mathbb{F}^{n,n} \).

**Solution:**

Since \( \text{Sym}(n) \subseteq \mathbb{F}^{n,n} \), we just need to show three things: (1) \( 0 \in \text{Sym}(n) \), (2) \( \text{Sym}(n) \) is closed under addition, (3) \( \text{Sym}(n) \) is closed under scalar multiplication.

- Clearly \( 0 \in \text{Sym}(n) \).
- Let \( A, B \in \text{Sym}(n) \). Then:
  \[
  (A + B)_{j,k} = A_{j,k} + B_{j,k} = A_{k,j} + B_{k,j} = (A + B)_{k,j}.
  \]
- Let \( A \in \text{Sym}(n) \) and \( \lambda \in \mathbb{F} \). Then:
  \[
  (\lambda A)_{j,k} = \lambda A_{j,k} = \lambda A_{k,j} = (\lambda A)_{k,j}.
  \]

(b) [6 pts] Write down a basis for \( \text{Sym}(3) \).

**Solution:**

First note that:

\[
\text{Sym}(3) = \left\{ \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{F} \right\}
\]

Therefore a basis for \( \text{Sym}(3) \) is:

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]

(c) [6 pts] What is the dimension of \( \text{Sym}(3) \)? Explain your answer.

**Solution:**

\( \dim \text{Sym}(3) = 6 \) since \( \#B = 6 \).

(d) [6 pts] Find a subspace \( U \) of \( \mathbb{F}^{2,2} \) such that \( \mathbb{F}^{2,2} = \text{Sym}(2) \oplus U \). For whatever \( U \) you write down, make sure your prove that \( \mathbb{F}^{2,2} = \text{Sym}(2) \oplus U \).

**Solution:**

First note that:

\[
\text{Sym}(2) = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : a, b, c \in \mathbb{F} \right\}.
\]
Now define $U$ as:

$$U = \left\{ \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} : d \in \mathbb{F} \right\}.$$

Clearly $\text{Sym}(2) \cap U = \{0\}$.

Now let $A \in \mathbb{F}^{2,2}$ with:

$$A = \begin{pmatrix} a & e \\ f & c \end{pmatrix}.$$

Then

$$A = \begin{pmatrix} a & e \\ f & c \end{pmatrix} = \begin{pmatrix} a & b \\ f & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix},$$

if and only if for every $e, f \in \mathbb{F}$ there exists $b, d \in \mathbb{F}$ such that

$$b + d = e,$$
$$b - d = f.$$

But this is just two equations and two unknowns, which we know is always solvable, so such a $b, d$ exist for every $e, f$. Therefore $\mathbb{F}^{2,2} = \text{Sym}(2) + U$ and so $\mathbb{F}^{2,2} = \text{Sym}(2) \oplus U$. 
**Question 3:**

The set $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a 2-tuple $(x, y) \in \mathbb{R}^2$.

(a) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as a complex vector space, show that the function

$$T : \mathbb{C} \to \mathbb{C}$$

$$T(z) = \alpha z$$

is a linear operator on $\mathbb{C}$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

**Solution:**

Need to show additivity and homogeneity to prove $T$ is linear:

- $T(z + w) = \alpha(z + w) = \alpha z + \alpha w = T(z) + T(w)$.
- $T(\lambda z) = \alpha(\lambda z) = \lambda(\alpha z) = \lambda T(z)$.

The matrix of $T$ is simply $\mathcal{M}(T) = (\alpha)$.

(b) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that the function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T(x + iy) = \alpha(x + iy)$$

is a linear operator on $\mathbb{R}^2$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

**Solution:**

First note that:

$$\alpha z = (a + ib)(x + iy) = ax - by + i(bx + ay).$$

Therefore, treating $\mathbb{C}$ as $\mathbb{R}^2$,

$$T(x, y) = (ax - by, bx + ay).$$

Now we show $T$ is linear:

- For additivity:

$$T(x + x', y + y') = (a(x + x') - b(y + y'), b(x + x') + a(y + y'))$$

$$= (ax + ax' - by - by', bx + bx' + ay + ay'),$$

$$= (ax - by, bx + ay) + (ax' - by', bx' + ay'),$$

$$= T(x, y) + T(x', y').$$

- For homogeneity:

$$T(\lambda x, \lambda y) = (a(\lambda x) - b(\lambda y), b(\lambda x) + a(\lambda y)) = \lambda(ax - by, bx + ay) = \lambda T(x, y).$$
The matrix of $T$ is:

$$\mathcal{M}(T) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$ 

(c) [8 pts] Define

$$T(x + iy) = 2x - y + i(x - 3y).$$

Show that this function is not a linear operator on the complex vector space $\mathbb{C}$, but it is a linear operator if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$.

**Solution:**

Showing $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear is essentially the same proof as the proof of linearity for part (b).

$T : \mathbb{C} \rightarrow \mathbb{C}$ is a function between one dimensional vector spaces. We know from the homework that $T$ is linear if and only if $T$ is scalar multiplication, i.e., $T(z) = \alpha z$ for some $\alpha \in \mathbb{C}$. But from part (b) we know that:

$$\alpha z = (a + ib)(x + iy) = ax - by + i(bx + ay),$$

and clearly $T$ does not have this form. Therefore $T$ cannot be linear.
Question 4:

Let $V$ be a finite dimensional complex vector space. An operator $T \in \mathcal{L}(V)$ is called nilpotent if $T^k = 0$ for some $k \in \mathbb{Z}$, $k \geq 1$, i.e., $T^k v = 0$ for all $v \in V$.

(a) [12 pts] What are the eigenvalues of a nilpotent operator? Make sure to prove your assertion.

Solution:
Let $\lambda \in \mathbb{C}$ be an eigenvalue with eigenvector $v \neq 0$. Then:

$$Tv = \lambda v \Rightarrow 0 = T^k v = \lambda^k v \Rightarrow \lambda^k = 0 \Rightarrow \lambda = 0.$$

Therefore the only eigenvalue of $T$ is 0.

(b) [12 pts] For a nilpotent operator $T$, let $k_0 \in \mathbb{Z}$, $k_0 \geq 1$, be the smallest positive integer such that $T^{k_0} = 0$. So in particular:

$$T^k \neq 0, \ \text{for all} \ 1 \leq k < k_0,$$

$$T^{k_0} = 0.$$

For which $k_0$ is $T$ diagonalizable and for which $k_0$ is $T$ not diagonalizable? Make sure to prove your assertion.

Solution:
If $k_0 = 1$ then $T = 0$ and $\mathcal{M}(T) = 0$, which is a diagonal matrix. So for $k_0 = 1$, $T$ is diagonalizable.

Now suppose $k_0 > 1$. Then $T \neq 0$. Now suppose $T$ is diagonalizable. We know that the diagonal entries of $\mathcal{M}(T)$ correspond to the eigenvalues of $T$. But by part (a) the only eigenvalue of $T$ is zero. Therefore $\mathcal{M}(T) = 0$. But this implies that $T = 0$, which is contradiction. Therefore $T$ is not diagonalizable.