Beginning of Lecture 26

The null space and range of T are related to the null space and range of T^* through the orthogonal complement, as we now prove.

Proposition 50. *If $T \in \mathcal{L}(V, W)$, then:*

1. $\operatorname{null} T^* = (\operatorname{range} T)^\perp$
2. $\operatorname{range} T^* = (\operatorname{null} T)^\perp$
3. $\operatorname{null} T = (\operatorname{range} T^*)^\perp$
4. $\operatorname{range} T = (\operatorname{null} T^*)^\perp$

Proof. We prove #1 first:

$$\begin{aligned} w \in \operatorname{null} T^* &\iff T^* w = 0 \\ &\iff \langle v, T^* w \rangle = 0, \ \ \forall\, v \in V \\ &\iff \langle Tv, w \rangle = 0, \ \ \forall\, v \in V \\ &\iff w \in (\operatorname{range} T)^\perp \end{aligned}$$

Thus $\operatorname{null} T^* = (\operatorname{range} T)^\perp$.

The rest now follow easily. Indeed, taking the orthogonal complement of both sides of #1 gives #4. Replacing T with T^* in #1 gives #3, and in number #4 gives #2. □

We now relate the adjoint to matrices.

Definition 48. The conjugate transpose of an $m \times n$ matrix $A \in \mathbb{F}^{m,n}$ is the $n \times m$ matrix $A^\dagger \in \mathbb{F}^{n,m}$ defined as:

$$A^\dagger_{j,k} = \overline{A}_{k,j}, \quad \forall\, j = 1, \ldots, n, \ \ k = 1, \ldots, m$$

Proposition 51. *Let $T \in \mathcal{L}(V, W)$, $\mathcal{B}_V = e_1, \ldots, e_n$ be an ONB of V, and $\mathcal{B}_W = f_1, \ldots, f_m$ be an ONB of W (note: they must be orthonormal!!). Then:*

$$\mathcal{M}(T^*; \mathcal{B}_W, \mathcal{B}_V) = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)^\dagger$$

Proof. Let $A = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)$. Recall that $A_{j,k}$ is defined by writing Te_k as a linear combinations of $f_1, \ldots, f_m$:

$$Te_k = \sum_{j=1}^{m} A_{j,k} f_j = \sum_{j=1}^{m} \langle Te_k, f_j \rangle_W f_j \Longrightarrow A_{j,k} = \langle Te_k, f_j \rangle_W$$

where the second equality follows since $\mathcal{B}_W$ is an ONB.

Now let $B = \mathcal{M}(T^*; \mathcal{B}_W, \mathcal{B}_V)$. Then B is defined as:

$$T^* f_k = \sum_{j=1}^{n} B_{j,k} e_j = \sum_{j=1}^{n} \langle T^* f_k, e_j \rangle_V e_j \Longrightarrow B_{j,k} = \langle T^* f_k, e_j \rangle_V$$

But then:

$$B_{j,k} = \langle T^* f_k, e_j \rangle = \overline{\langle e_j, T^* f_k \rangle} = \overline{\langle Te_j, f_k \rangle} = \overline{A}_{k,j} = A^{\dagger}_{j,k}$$

□

Now we focus in on operators $T \in \mathcal{L}(V)$, where V is an inner product space. We shall be particularly interested in the following operators.

Definition 49. An operator $T \in \mathcal{L}(V)$ is self-adjoint if $T = T^*$, i.e.,

$$\langle Tv, w \rangle = \langle v, Tw \rangle, \quad \forall\, v, w \in V$$

Remark: The previous proposition shows that for a general $T \in \mathcal{L}(V)$, if $\mathcal{B}$ is an ONB for V, then $\mathcal{M}(T^*; \mathcal{B}) = \mathcal{M}(T; \mathcal{B})^{\dagger}$. But if T is self-adjoint, then $T = T^*$ and so $\mathcal{M}(T; \mathcal{B}) = \mathcal{M}(T^*; \mathcal{B}) = \mathcal{M}(T; \mathcal{B})^{\dagger}$, which implies that $\mathcal{M}(T; \mathcal{B})$ is symmetric and real valued.

Proposition 52. *The eigenvalues of self-adjoint operators are real valued (even when $\mathbb{F} = \mathbb{C}$).*

Proof. Let $T \in \mathcal{L}(V)$ be self-adjoint, $\lambda \in \mathbb{F}$ and eigenvalue of T, and $v \in V$ a corresponding nonzero eigenvector so that $Tv = \lambda v$. Then:

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$$

□

Proposition 53. *Let V be a complex inner product space and $T \in \mathcal{L}(V)$.*

$$\text{If } \langle Tv, v\rangle = 0,\ \ \forall\, v \in V,\ \text{ then } T = 0$$

Proof. Suppose $\langle Tv, v\rangle = 0,\ \forall\, v \in V$. Let $u, w \in V$ and consider the clever rewriting of $\langle Tu, w\rangle$:

$$\begin{aligned}
\langle Tu, w\rangle &= \frac{1}{4}\langle T(u+w), \underbrace{u+w}_{v_1}\rangle - \frac{1}{4}\langle T(u-w), \underbrace{u-w}_{v_2}\rangle \\
&\quad + \frac{1}{4}\langle T(u+iw), \underbrace{u+iw}_{v_3}\rangle i - \frac{1}{4}\langle T(u-iw), \underbrace{u-iw}_{v_4}\rangle i \\
&= \frac{1}{4}\left(\langle Tv_1, v_1\rangle + \langle Tv_2, v_2\rangle + \langle Tv_3, v_3\rangle i + \langle Tv_4, v_4\rangle i\right) \\
&= 0
\end{aligned}$$

Thus $\langle Tu, w\rangle = 0,\ \forall\, u, w \in V$. Taking $w = Tu$, we get $\|Tu\|^2 = 0,\ \forall\, u \in V$, which implies that $Tu = 0$ for all $u \in V$, and so $T = 0$. □

Remark: False if $\mathbb{F} = \mathbb{R}$. Take $V = \mathbb{R}^2$ and T to be a 90-degree rotation.

Proposition 54. *Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then:*

$$T \text{ is self-adjoint} \Longleftrightarrow \langle Tv, v\rangle \in \mathbb{R},\ \ \forall\, v \in V$$

Proof. Let $v \in V$, then:

$$\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = \langle Tv, v\rangle - \langle v, Tv\rangle = \langle Tv, v - \langle T^*v, v\rangle = \langle (T-T^*)v, v\rangle \quad (15)$$

If $\langle Tv, v\rangle \in \mathbb{R}$, then by (15):

$$0 = \langle (T-T^*)v, v\rangle \Longrightarrow T - T^* = 0 \text{ [by previous Proposition] } \Longrightarrow T = T^*$$

Conversely, if T is self=adjoint then (15) also implies:

$$\langle Tv, v\rangle - \overline{\langle Tv, v\rangle} = 0 \Longrightarrow \langle Tv, v\rangle = \overline{\langle Tv, v\rangle} \Longrightarrow \langle Tv, v\rangle \in \mathbb{R}$$

□

Remark: Also false if $\mathbb{F} = \mathbb{R}$ since $\langle Tv, v\rangle \in \mathbb{R}$ for all $T \in \mathcal{L}(V)$, including those that are not self-adjoint.

End of Lecture 26