BEGINNING OF LECTURE 26

The null space and range of $T$ are related to the null space and range of $T^*$ through the orthogonal complement, as we now prove.

**Proposition 50.** If $T \in \mathcal{L}(V, W)$, then:

1. $\text{null } T^* = (\text{range } T)^\perp$
2. $\text{range } T^* = (\text{null } T)^\perp$
3. $\text{null } T = (\text{range } T^*)^\perp$
4. $\text{range } T = (\text{null } T^*)^\perp$

**Proof.** We prove #1 first:

$$w \in \text{null } T^* \iff T^* w = 0 \iff \langle v, T^* w \rangle = 0, \quad \forall v \in V \iff \langle Tv, w \rangle = 0, \quad \forall v \in V \iff w \in (\text{range } T)^\perp$$

Thus $\text{null } T^* = (\text{range } T)^\perp$.

The rest now follow easily. Indeed, taking the orthogonal complement of both sides of #1 gives #4. Replacing $T$ with $T^*$ in #1 gives #3, and in number #4 gives #2.

We now relate the adjoint to matrices.

**Definition 48.** The conjugate transpose of an $m \times n$ matrix $A \in \mathbb{F}^{m,n}$ is the $n \times m$ matrix $A^\dagger \in \mathbb{F}^{n,m}$ defined as:

$$A^\dagger_{j,k} = \overline{A}_{k,j}, \quad \forall j = 1, \ldots, n, \quad k = 1, \ldots, m$$

**Proposition 51.** Let $T \in \mathcal{L}(V, W)$, $\mathcal{B}_V = e_1, \ldots, e_n$ be an ONB of $V$, and $\mathcal{B}_W = f_1, \ldots, f_m$ be an ONB of $W$ (note: they must be orthonormal!!). Then:

$$\mathcal{M}(T^*; \mathcal{B}_W, \mathcal{B}_V) = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)^\dagger$$
Proof. Let \( A = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W) \). Recall that \( A_{j,k} \) is defined by writing \( Te_k \) as a linear combinations of \( f_1, \ldots, f_m \):

\[
Te_k = \sum_{j=1}^{m} A_{j,k} f_j = \sum_{j=1}^{m} \langle Te_k, f_j \rangle w f_j \quad \Rightarrow \quad A_{j,k} = \langle Te_k, f_j \rangle w
\]

where the second equality follows since \( \mathcal{B}_W \) is an ONB.

Now let \( B = \mathcal{M}(T^*; \mathcal{B}_W, \mathcal{B}_V) \). Then \( B \) is defined as:

\[
T^* f_k = \sum_{j=1}^{n} B_{j,k} e_j = \sum_{j=1}^{n} \langle T^* f_k, e_j \rangle V e_j \quad \Rightarrow \quad B_{j,k} = \langle T^* f_k, e_j \rangle V
\]

But then:

\[
B_{j,k} = \langle T^* f_k, e_j \rangle = \overline{\langle e_j, T^* f_k \rangle} = \overline{\langle Te_j, f_k \rangle} = \overline{A_{k,j}} = A_{j,k}^\dagger
\]

Now we focus in on operators \( T \in \mathcal{L}(V) \), where \( V \) is an inner product space. We shall be particularly interested in the following operators.

**Definition 49.** An operator \( T \in \mathcal{L}(V) \) is **self-adjoint** if \( T = T^* \), i.e.,

\[
\langle Tv, w \rangle = \langle v, Tw \rangle, \quad \forall v, w \in V
\]

**Remark:** The previous proposition shows that for a general \( T \in \mathcal{L}(V) \), if \( \mathcal{B} \) is an ONB for \( V \), then \( \mathcal{M}(T^*; \mathcal{B}) = \mathcal{M}(T; \mathcal{B})^\dagger \). But if \( T \) is self-adjoint, then \( T = T^* \) and so \( \mathcal{M}(T; \mathcal{B}) = \mathcal{M}(T^*; \mathcal{B}) = \mathcal{M}(T; \mathcal{B})^\dagger \), which implies that \( \mathcal{M}(T; \mathcal{B}) \) is symmetric and real valued.

**Proposition 52.** The eigenvalues of self-adjoint operators are real valued (even when \( F = \mathbb{C} \)).

**Proof.** Let \( T \in \mathcal{L}(V) \) be self-adjoint, \( \lambda \in \mathbb{F} \) and eigenvalue of \( T \), and \( v \in V \) a corresponding nonzero eigenvector so that \( Tv = \lambda v \). Then:

\[
\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \|v\|^2 \Rightarrow \lambda = \overline{\lambda} \Rightarrow \lambda \in \mathbb{R}
\]
Proposition 53. Let \( V \) be a complex inner product space and \( T \in \mathcal{L}(V) \).

If \( \langle Tv, v \rangle = 0, \ \forall v \in V \), then \( T = 0 \)

Proof. Suppose \( \langle Tv, v \rangle = 0, \ \forall v \in V \). Let \( u, w \in V \) and consider the clever rewriting of \( \langle Tu, w \rangle \):

\[
\langle Tu, w \rangle = \frac{1}{4} \langle T(u + w), u + w \rangle - \frac{1}{4} \langle T(u - w), u - w \rangle \\
+ \frac{1}{4} \langle T(u + iw), u + iw \rangle i - \frac{1}{4} \langle T(u - iw), u - iw \rangle i \\
= \frac{1}{4} (\langle Tv_1, v_1 \rangle + \langle Tv_2, v_2 \rangle + \langle Tv_3, v_3 \rangle i + \langle Tv_4, v_4 \rangle i) \\
= 0
\]

Thus \( \langle Tu, w \rangle = 0, \ \forall u, w \in V \). Taking \( w = Tu \), we get \( \|Tu\|^2 = 0, \ \forall u \in V \), which implies that \( Tu = 0 \) for all \( u \in V \), and so \( T = 0 \). \( \square \)

Remark: False if \( \mathbb{F} = \mathbb{R} \). Take \( V = \mathbb{R}^2 \) and \( T \) to be a 90-degree rotation.

Proposition 54. Suppose \( V \) is a complex inner product space and \( T \in \mathcal{L}(V) \). Then:

\( T \) is self-adjoint \( \iff \langle Tv, v \rangle \in \mathbb{R}, \ \forall v \in V \)

Proof. Let \( v \in V \), then:

\[
\langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = \langle Tv, v \rangle - \langle v, Tv \rangle = \langle Tv, v - \langle T^*v, v \rangle \rangle = \langle (T - T^*)v, v \rangle \quad (15)
\]

If \( \langle Tv, v \rangle \in \mathbb{R} \), then by (15):

\[
0 = \langle (T - T^*)v, v \rangle \implies T - T^* = 0 \ [\text{by previous Proposition}] \implies T = T^*
\]

Conversely, if \( T \) is self-adjoint then (15) also implies:

\[
\langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = 0 \implies \langle Tv, v \rangle = \overline{\langle Tv, v \rangle} \implies \langle Tv, v \rangle \in \mathbb{R}
\]

Remark: Also false if \( \mathbb{F} = \mathbb{R} \) since \( \langle Tv, v \rangle \in \mathbb{R} \) for all \( T \in \mathcal{L}(V) \), including those that are not self-adjoint.

End of Lecture 26