BEGINNING OF LECTURE 27

Proposition 55. If \( T \) is self-adjoint and \( \langle Tv, v \rangle = 0 \) for all \( v \in V \), then \( T = 0 \) (even if \( \mathbb{F} = \mathbb{R} \)).

Proof. If \( \mathbb{F} = \mathbb{C} \) then we already proved this. So assume that \( \mathbb{F} = \mathbb{R} \), and suppose that \( T \) is self-adjoint and \( \langle Tv, v \rangle = 0 \) for all \( v \in V \). Let \( u, w \in V \); then:

\[
\langle T(u + w), u + w \rangle - \langle T(u - w), u - w \rangle = \\
= \langle Tu, u \rangle + \langle Tu, w \rangle + \langle Tw, u \rangle + \langle Tw, w \rangle - \langle Tu, u \rangle - \langle Tu, w \rangle - \langle Tw, u \rangle - \langle Tw, w \rangle \\
= 2\langle Tu, w \rangle + 2\langle Tw, u \rangle \\
= 2\langle Tu, w \rangle + 2\langle u, Tw \rangle \quad [\mathbb{F} = \mathbb{R}] \\
= 4\langle Tu, w \rangle \quad [T = T^*]
\]

Thus, let \( v_1 = u + w \) and \( v_2 = u - w \):

\[
\langle Tu, w \rangle = \frac{1}{4}(\langle Tv_1, v_1 \rangle + \langle Tv_2, v_2 \rangle) = 0, \quad \forall u, w \in V \implies T = 0
\]

Self-adjoint operators are a subset of the following class of important operators.

Definition 50. \( T \in \mathcal{L}(V) \) is normal if

\[
TT^* = T^*T
\]

Example: Recall the vector space:

\[
V = \{ f \in C^\infty([0, 1]; \mathbb{R}) : f^{(k)}(0) = f^{(k)}(1), \quad \forall k = 0, 1, 2, \ldots \}
\]

and the differentiation operator \( D \in \mathcal{L}(V) \),

\[
Df = f'
\]

We showed that \( D^* = -D \). Thus \( D \) is not self adjoint, but

\[
DD^* = D(-D) = -D^2 = (-D)D = D^*D
\]

and so \( D \) is normal.
Proposition 56.

\[ T \text{ is normal } \iff \|Tv\| = \|T^*v\|, \ \forall v \in V \]

Proof. We have:

\[ T \text{ is normal } \iff T^*T - TT^* = 0 \]
\[ \iff \langle (T^*T - TT^*)v, v \rangle = 0, \ \forall v \in V \quad (16) \]
\[ \iff \langle T^*Tv, v \rangle = \langle TT^*v, v \rangle, \ \forall v \in V \]
\[ \iff \|Tv\|^2 = \|T^*v\|^2, \ \forall v \in V \]

where (16) follows from the previous Proposition since \( T^*T - TT^* \) is self-adjoint.

Corollary 8. Suppose \( T \in \mathcal{L}(V) \) is normal. If \( v \in V \) is an eigenvector of \( T \) with eigenvalue \( \lambda \), then \( v \) is an eigenvector of \( T^* \) with eigenvalue \( \bar{\lambda} \).

Proof. \( T \) normal implies that \( T - \lambda I \) is normal since:

\[
(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda}I)
\]
\[
= TT^* - \bar{\lambda}T - \lambda T^* + \lvert \lambda \rvert^2 I
\]
\[
= T^*T - \bar{\lambda}T - \lambda T + \lvert \lambda \rvert^2 I
\]
\[
= (T^* - \bar{\lambda}I)(T - \lambda I)
\]
\[
= (T - \lambda I)^*(T - \lambda I)
\]

Then by the previous Proposition:

\[
0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^*v\| = \|(T^* - \bar{\lambda}I)v\| \implies T^*v = \bar{\lambda}v
\]

Proposition 57. If \( T \) is normal, then the eigenvectors of \( T \) corresponding to distinct eigenvalues are orthogonal.

Proof. Let \( \alpha, \beta \) be distinct eigenvalues of \( T \) with corresponding eigenvectors \( u, v \) so that

\[
Tu = \alpha u \quad \text{and} \quad Tv = \beta v
\]

From the previous Corollary we have \( T^*v = \bar{\beta}v \). Thus:

\[
(\alpha - \beta)\langle u, v \rangle = \alpha \langle u, v \rangle - \beta \langle u, v \rangle = \langle \alpha u, v \rangle - \langle u, \bar{\beta}v \rangle = \langle Tu, v \rangle - \langle u, T^*v \rangle = 0
\]

Since \( \alpha \neq \beta \), we must have \( \langle u, v \rangle = 0 \).
7.B The Spectral Theorem

Two flavors, real and complex. As is often the case, the complex version is in fact easier. So we start with that.

Complex Spectral Theorem

**Theorem 25** (Complex Spectral Theorem). Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. $T$ is normal
2. $V$ has an ONB consisting of eigenvectors of $T$
3. $T$ has a diagonal matrix with respect to some ONB of $V$

End of Lecture 27