Beginning of Lecture 27

Proposition 55. *If T is self-adjoint and $\langle Tv, v\rangle = 0$ for all $v \in V$, then $T = 0$ (even if $\mathbb{F} = \mathbb{R}$).*

Proof. If $\mathbb{F} = \mathbb{C}$ then we already proved this. So assume that $\mathbb{F} = \mathbb{R}$, and suppose that T is self-adjoint and $\langle Tv, v\rangle = 0$ for all $v \in V$. Let $u, w \in V$; then:

$$\begin{aligned}
&\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle = \\
&= \langle Tu, u\rangle + \langle Tu, w\rangle + \langle Tw, u\rangle + \langle Tw, w\rangle - \langle Tu, u\rangle + \langle Tu, w\rangle + \langle Tw, u\rangle - \langle Tw, w\rangle \\
&= 2\langle Tu, w\rangle + 2\langle Tw, u\rangle \\
&= 2\langle Tu, w\rangle + 2\langle u, Tw\rangle \ \ [\mathbb{F} = \mathbb{R}] \\
&= 4\langle Tu, w\rangle \ \ [T = T^*]
\end{aligned}$$

Thus, let $v_1 = u + w$ and $v_2 = u - w$:

$$\langle Tu, w\rangle = \frac{1}{4}(\langle Tv_1, v_1\rangle + \langle Tv_2, v_2\rangle) = 0, \ \ \forall\, u, w \in V \Longrightarrow T = 0$$

□

Self-adjoint operators are a subset of the following class of important operators.

Definition 50. $T \in \mathcal{L}(V)$ is <u>normal</u> if

$$TT^* = T^*T$$

<u>Example:</u> Recall the vector space:

$$V = \{f \in C^\infty([0,1]; \mathbb{R}) : f^{(k)}(0) = f^{(k)}(1), \ \ \forall\, k = 0, 1, 2, \ldots\}$$

and the differentiation operator $D \in \mathcal{L}(V)$,

$$Df = f'$$

We showed that $D^* = -D$. Thus D is not self adjoint, but

$$DD^* = D(-D) = -D^2 = (-D)D = D^*D$$

and so D is normal.

Proposition 56.

$$T \text{ is normal } \iff \|Tv\| = \|T^*v\|, \ \forall\, v \in V$$

Proof. We have:

$$\begin{aligned} T \text{ is normal } &\iff T^*T - TT^* = 0 \\ &\iff \langle (T^*T - TT^*)v, v\rangle = 0, \ \forall\, v \in V \qquad (16) \\ &\iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle, \ \forall\, v \in V \\ &\iff \|Tv\|^2 = \|T^*v\|^2, \ \forall\, v \in V \end{aligned}$$

where (16) follows from the previous Proposition since $T^*T - TT^*$ is self-adjoint. □

Corollary 8. *Suppose $T \in \mathcal{L}(V)$ is normal. If $v \in V$ is an eigenvector of T with eigenvalue λ, then v is an eigenvector of T^* with eigenvalue $\bar{\lambda}$.*

Proof. T normal implies that $T - \lambda I$ is normal since:

$$\begin{aligned} (T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \bar{\lambda} I) \\ &= TT^* - \bar{\lambda} T - \lambda T^* + |\lambda|^2 I \\ &= T^*T - \lambda T^* - \bar{\lambda} T + |\lambda|^2 I \\ &= (T^* - \bar{\lambda} I)(T - \lambda I) \\ &= (T - \lambda I)^*(T - \lambda I) \end{aligned}$$

Then by the previous Proposition:

$$0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^* v\| = \|(T^* - \bar{\lambda} I)v\| \implies T^*v = \bar{\lambda} v$$

□

Proposition 57. *If T is normal, then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Proof. Let α, β be distinct eigenvalues of T with corresponding eigenvectors u, v so that

$$Tu = \alpha u \qquad \text{and} \qquad Tv = \beta v$$

From the previous Corollary we have $T^*v = \bar{\beta} v$. Thus:

$$(\alpha - \beta)\langle u, v\rangle = \alpha\langle u, v\rangle - \beta\langle u, v\rangle = \langle \alpha u, v\rangle - \langle u, \bar{\beta} v\rangle = \langle Tu, v\rangle - \langle u, T^*v\rangle = 0$$

Since $\alpha \neq \beta$, we must have $\langle u, v\rangle = 0$. □

7.B The Spectral Theorem

Two flavors, real and complex. As is often the case, the complex version is in fact easier. So we start with that.

Complex Spectral Theorem

Theorem 25 (Complex Spectral Theorem). *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$. *Then the following are equivalent:*

1. *T is normal*
2. *V has an ONB consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some ONB of V*

End of Lecture 27