Beginning of Lecture 28

Theorem 26 (Complex Spectral Theorem)**.** *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$*. Then the following are equivalent:*

1. *T is normal*
2. *V has an ONB consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some ONB of V*

Proof. We prove this in parts:

- (2) $\iff$ (3) follows from our work in Chapter 5.
- (3) $\implies$ (1): Let $\mathcal{B}$ be an ONB such that $\mathcal{M}(T;\mathcal{B})$ is diagonal. Then

$$\mathcal{M}(T^*;\mathcal{B}) = \mathcal{M}(T;\mathcal{B})^\dagger \implies \mathcal{M}(T^*;\mathcal{B}) \text{ is diagonal}$$

  But then since diagonal matrices commute we have:

$$\begin{aligned}\mathcal{M}(TT^*;\mathcal{B}) = \mathcal{M}(T;\mathcal{B})\mathcal{M}(T^*;\mathcal{B}) = \mathcal{M}(T^*;\mathcal{B})\mathcal{M}(T;\mathcal{B}) &= \mathcal{M}(T^*T;\mathcal{B}) \\ \implies TT^* &= T^*T\end{aligned}$$

  since $\mathcal{L}(V)$ and $\mathbb{C}^{n,n}$ are isomorphic under the linear map $\mathcal{M} : \mathcal{L}(V) \to \mathbb{C}^{n,n}$.

- Now suppose that T is normal. By Schur's Theorem there exists and ONB $\mathcal{B} = e_1, \ldots, e_n$ such that $\mathcal{M}(T;\mathcal{B})$ is upper triangular; write the matrix as:

$$\mathcal{M}(T;\mathcal{B}) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix} \implies \mathcal{M}(T^*;\mathcal{B}) = \mathcal{M}(T;\mathcal{B})^\dagger = \begin{pmatrix} \bar{a}_{1,1} & & 0 \\ \vdots & \ddots & \\ \bar{a}_{1,n} & \cdots & \bar{a}_{n,n} \end{pmatrix}$$

  We now show that $\mathcal{M}(T;\mathcal{B})$ is in fact a diagonal matrix. Indeed, because $\mathcal{B}$ is an ONB and T is normal:

$$\begin{aligned} \|Te_1\|^2 &= |a_{1,1}|^2 \\ \|Te_1\|^2 &= \|T^*e_1\|^2 = |a_{1,1}|^2 + |a_{1,2}|^2 + \cdots + |a_{1,n}|^2 \\ &\implies a_{1,2} = \cdots = a_{1,n} = 0 \end{aligned}$$

Now we also have:

$$\begin{aligned}\|Te_2\|^2 &= |a_{1,2}|^2 + |a_{2,2}|^2 = 0 + |a_{2,2}|^2 = |a_{2,2}|^2 \\ \|Te_2\|^2 &= \|T^*e_2\|^2 = |a_{2,2}|^2 + |a_{2,3}|^2 + \cdots |a_{2,n}|^2 \\ &\implies a_{2,3} = \cdots = a_{2,n} = 0\end{aligned}$$

Continuing in this fashion we see that all off-diagonal entries must be zero.

□

Next week we will prove the Real Spectral Theorem:

Theorem 27 (Real Spectral Theorem)**.** *Suppose* $\mathbb{F} = \mathbb{R}$ *and* $T \in \mathcal{L}(V)$. *Then the following are equivalent:*

1. *T is self-adjoint*
2. *V has an ONB consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some ONB of V*

End of Lecture 28