Beginning of Lecture 29

Warmup: If we change $\mathbb{F} = \mathbb{R}$, where does the proof of the Complex Spectral Theorem fall apart?

Answer: To prove (1) $\Longrightarrow$ (3) we used Schur's Theorem, which only applies to complex vector spaces.

Real Spectral Theorem

We now aim to prove the Real Spectral Theorem:

Theorem 28 (Real Spectral Theorem)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:*

1. *T is self-adjoint*
2. *V has an ONB consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some ONB of V*

The Real Spectral Theorem is harder to prove and as such we will first need some preliminary results.

Consider the quadratic polynomial $p \in \mathcal{P}_2(\mathbb{R})$:

$$p(x) = x^2 + bx + c, \quad x, b, c \in \mathbb{R}$$

Note the following:

$$\text{If } b^2 < 4c, \text{ then } x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) > 0, \ \forall\, x \in \mathbb{R}$$

In particular $p(x) > 0$ so it has a multiplicative inverse for all $x \in \mathbb{R}$, namely $p(x) \cdot (1/p(x)) = 1$. A similar type of reasoning leads to the following result.

Proposition 58. *If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ satisfy $b^2 < 4c$, then*

$$p(T) = T^2 + bT + cI$$

is invertible.

Proof. Let $v \in V$, $v \neq 0$. Then:

$$\begin{aligned}
\langle p(T)v, v\rangle &= \langle (T^2 + bT + cI)v, v\rangle \\
&= \langle T^2 v, v\rangle + b\langle Tv, v\rangle + c\langle v, v\rangle \\
&= \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c\|v\|^2 \\
&\geq \|Tv\|^2 - |b|\|Tv\|\|v\| + c\|v\|^2 \text{ [Cauchy-Schwarz]} \\
&= \left(\|Tv\| - \frac{|b|\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 \\
&> 0
\end{aligned}$$

Thus $p(T)v \neq 0 \Longrightarrow p(T)$ is injective, and hence invertible. □

End of Lecture 29