BEGINNING OF LECTURE 29

Warmup: If we change \( F = \mathbb{R} \), where does the proof of the Complex Spectral Theorem fall apart?

Answer: To prove \((1) \implies (3)\) we used Schur’s Theorem, which only applies to complex vector spaces.

Real Spectral Theorem

We now aim to prove the Real Spectral Theorem:

**Theorem 28** (Real Spectral Theorem). Suppose \( F = \mathbb{R} \) and \( T \in \mathcal{L}(V) \). Then the following are equivalent:

1. \( T \) is self-adjoint
2. \( V \) has an ONB consisting of eigenvectors of \( T \)
3. \( T \) has a diagonal matrix with respect to some ONB of \( V \)

The Real Spectral Theorem is harder to prove and as such we will first need some preliminary results.

Consider the quadratic polynomial \( p \in \mathcal{P}_2(\mathbb{R}) \):

\[
p(x) = x^2 + bx + c, \quad x, b, c \in \mathbb{R}
\]

Note the following:

If \( b^2 < 4c \), then

\[
x^2 + bx + c = \left( x + \frac{b}{2} \right)^2 + \left( c - \frac{b^2}{4} \right) > 0, \quad \forall x \in \mathbb{R}
\]

In particular \( p(x) > 0 \) so it has a multiplicative inverse for all \( x \in \mathbb{R} \), namely \( p(x) \cdot (1/p(x)) = 1 \). A similar type of reasoning leads to the following result.

**Proposition 58.** If \( T \in \mathcal{L}(V) \) is self-adjoint and \( b, c \in \mathbb{R} \) satisfy \( b^2 < 4c \), then

\[
p(T) = T^2 + bT + cI
\]

is invertible.
Proof. Let \( v \in V, v \neq 0 \). Then:

\[
\langle p(T)v, v \rangle = \langle (T^2 + bT + cI)v, v \rangle \\
= \langle T^2v, v \rangle + b \langle Tv, v \rangle + c \langle v, v \rangle \\
= \langle Tv, Tv \rangle + b \langle Tv, v \rangle + c \|v\|^2 \\
\geq \|Tv\|^2 - |b|\|Tv\|\|v\| + c \|v\|^2 \quad \text{[Cauchy-Schwarz]} \\
= \left(\|Tv\| - \frac{|b|\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 \\
> 0
\]

Thus \( p(T)v \neq 0 \implies p(T) \) is injective, and hence invertible. \( \square \)

End of Lecture 29