Beginning of Lecture 30

Proposition 59. If $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is self-adjoint, then $T$ has an eigenvalue (even if $\mathbb{F} = \mathbb{R}$).

To prove this, we are going to need the following proposition from Chapter 4 on polynomials.

Proposition 60. If $p \in \mathcal{P}(\mathbb{R})$ is a non-constant polynomial, then $p$ has a unique factorization (except for re-ordering) of the form:

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1x + c_1) \cdots (x^2 + b_Mx + c_M),$$

where

$$m + M \geq 1$$
$$c \in \mathbb{R}, c \neq 0$$
$$\lambda_1, \ldots, \lambda_m \in \mathbb{R}$$
$$b_1, \ldots, b_M \in \mathbb{R}$$
$$c_1, \ldots, c_M \in \mathbb{R}$$
$$b_j^2 < 4c_j, \ \forall j$$

Proof of Proposition 59. If $\mathbb{F} = \mathbb{C}$ then $T$ has an eigenvalue even if it is not self-adjoint (recall this from Chapter 5), so we can assume that $\mathbb{F} = \mathbb{R}$.

Let $n = \dim V$ and choose any $v \in V$ with $v \neq 0$; then:

$$v, Tv, T^2v, \ldots, T^nv$$
must be linearly dependent

Thus there exists $a_0, \ldots, a_n \in \mathbb{R}$, not all zero, such that

$$0 = \sum_{k=0}^{n} a_k T^k v$$

Define $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then using Proposition 60 we have:

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$
$$= c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1x + c_1) \cdots (x^2 + b_Mx + c_M)$$
Thus:

\[ 0 = a_0v + a_1Tv + \cdots + a_nT^n v \]
\[ = (a_0 + a_1T + \cdots + a_nT^n)v \]
\[ = c(T - \lambda_1I) \cdots (T - \lambda_mI)(T^2 + b_1T + c_1I) \cdots (T^2 + b_MT + c_MI)v \quad (17) \]

By Proposition 58, each \( T^2 + b_jT + c_jI \) is invertible. Also, \( c \neq 0 \), so \( m > 0 \) since otherwise the RHS of (17) would be an invertible (hence injective) operator acting a nonzero vector \( v \), but the LHS is zero, and thus we would have a contradiction. Therefore,

\[ 0 = (T - \lambda_1I) \cdots (T - \lambda_mI)v \implies T - \lambda_jI \text{ is not injective for some } j \]
\[ \implies \lambda_j \text{ is an eigenvalue of } T \]

\[ \square \]

**Proposition 61.** If \( T \in \mathcal{L}(V) \) is self-adjoint and \( U \) is an invariant subspace of \( V \) under \( T \), then:

1. \( U^\perp \) is invariant under \( T \)
2. \( T|_U \in \mathcal{L}(U) \) is self-adjoint
3. \( T|_{U^\perp} \in \mathcal{L}(U^\perp) \) is self-adjoint

**Proof.** We prove each part:

1. Let \( v \in U^\perp \) and \( u \in U \); then:

\[ \langle Tv, u \rangle = \langle v, \underbrace{Tu}_{\in U} \rangle = 0, \quad \forall u \in U \implies Tv \in U^\perp \]

Thus \( U^\perp \) is invariant under \( T \).

2. If \( u, v \in U \), then:

\[ \langle (T|_U)u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, (T|_U)v \rangle \implies (T|_U)^* = T|_U \]

3. Replace \( U \) with \( U^\perp \) in the proof of #2. This is valid because we have already proved #1.

\[ \square \]
**Theorem 29** (Real Spectral Theorem). Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. $T$ is self-adjoint
2. $V$ has an ONB consisting of eigenvectors of $T$
3. $T$ has a diagonal matrix with respect to some ONB of $V$

**Proof.** We prove $(1) \implies (2) \implies (3) \implies (1)$ in parts:

- $(2) \implies (3)$ is clear

- $(3) \implies (1)$: Let $\mathcal{B}$ be an ONB such that $\mathcal{M}(T; \mathcal{B}) \in \mathbb{R}^{n,n}$ is diagonal. Then $\mathcal{M}(T^*; \mathcal{B}) = \mathcal{M}(T; \mathcal{B})^\dagger = \mathcal{M}(T; \mathcal{B})$, and so we must have $T^* = T$.

- $(1) \implies (2)$: Proof by induction on $\dim V$. For the base case, let $\dim V = 1$. Since $T$ is guaranteed to have one eigenvalue, it has an eigenvector $v$ and necessarily $\text{span}(v) = V$.

Now suppose that $\dim V = n > 1$ and that $(1) \implies (2)$ for all vector spaces $U$ with $\dim U \leq n - 1$ and all self-adjoint $S \in \mathcal{L}(U)$. Let $T \in \mathcal{L}(V)$ be self-adjoint. Let $u \in V$ be an eigenvector of $T$ with $\|u\| = 1$, and let $U = \text{span}(u)$. Then $U$ is a 1-dimensional subspace of $V$ that is invariant under $T$. Thus $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

By the induction hypothesis, there exists an ONB $\mathcal{B}_\perp = u_1, \ldots, u_{n-1}$ of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. But then $\mathcal{B} = u_1, \ldots, u_{n-1}, u$ is an ONB of $V$ consisting of eigenvectors of $T$.

\[ \square \]

**End of Lecture 30**