Beginning of Lecture 30

Proposition 59. *If $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is self-adjoint, then T has an eigenvalue (even if $\mathbb{F} = \mathbb{R}$).*

To prove this, we are going to need the following proposition from Chapter 4 on polynomials.

Proposition 60. *If $p \in \mathcal{P}(\mathbb{R})$ is a non-constant polynomial, then p has a unique factorization (except for re-ordering) of the form:*

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where

$$\begin{aligned} m + M &\geq 1 \\ c &\in \mathbb{R}, c \neq 0 \\ \lambda_1, \ldots, \lambda_m &\in \mathbb{R} \\ b_1, \ldots, b_M &\in \mathbb{R} \\ c_1, \ldots, c_M &\in \mathbb{R} \\ b_j^2 &< 4c_j, \ \ \forall j \end{aligned}$$

Proof of Proposition 59. If $\mathbb{F} = \mathbb{C}$ then T has an eigenvalue even if it is not self-adjoint (recall this from Chapter 5), so we can assume that $\mathbb{F} = \mathbb{R}$.

Let $n = dimV$ and choose any $v \in V$ with $v \neq 0$; then:

$$v, Tv, T^2 v, \ldots, T^n v \text{ must be linearly dependent}$$

Thus there exists $a_0, \ldots, a_n \in \mathbb{R}$, not all zero, such that

$$0 = \sum_{k=0}^{n} a_k T^k v$$

Define $p(x) = a_0 + a_1 x + \cdots a_n x^n$. Then using Proposition 60 we have:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ &= c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M) \end{aligned}$$

Thus:

$$\begin{aligned}0 &= a_0 v + a_1 Tv + \cdots a_n T^n v \\ &= (a_0 + a_1 T + \cdots a_n T^n)v \\ &= c(T - \lambda_1 I)\cdots(T - \lambda_m I)(T^2 + b_1 T + c_1 I)\cdots(T^2 + b_M T + c_M I)v \quad (17)\end{aligned}$$

By Proposition 58, each $T^2 + b_j T + c_j I$ is invertible. Also, $c \neq 0$, so $m > 0$ since otherwise the RHS of (17) would be an invertible (hence injective) operator acting a nonzero vector v, but the LHS is zero, and thus we would have a contradiction. Therefore,

$$\begin{aligned}0 = (T - \lambda_1 I)\cdots(T - \lambda_m I)v &\implies T - \lambda_j I \text{ is not injective for some } j \\ &\implies \lambda_j \text{ is an eigenvalue of } T\end{aligned}$$

□

Proposition 61. *If $T \in \mathcal{L}(V)$ is self-adjoint and U is an invariant subspace of V under T, then:*

1. *$U^\perp$ is invariant under T*
2. *$T|_U \in \mathcal{L}(U)$ is self-adjoint*
3. *$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint*

Proof. We prove each part:

1. Let $v \in U^\perp$ and $u \in U$; then:

$$\langle Tv, u\rangle = \langle v, \underbrace{Tu}_{\in U}\rangle = 0, \ \ \forall\, u \in U \implies Tv \in U^\perp$$

Thus $U^\perp$ is invariant under T.

2. If $u, v \in U$, then:

$$\langle (T|_U)u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \langle u, (T|_U)v\rangle \implies (T|_U)^* = T|_U$$

3. Replace U with $U^\perp$ in the proof of #2. This is valid because we have already proved #1.

□

Theorem 29 (Real Spectral Theorem)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:*

1. *T is self-adjoint*
2. *V has an ONB consisting of eigenvectors of T*
3. *T has a diagonal matrix with respect to some ONB of V*

Proof. We prove (1) $\implies$ (2) $\implies$ (3) $\implies$ (1) in parts:

- (2) $\implies$ (3) is clear
- (3) $\implies$ (1): Let $\mathcal{B}$ be an ONB such that $\mathcal{M}(T;\mathcal{B}) \in \mathbb{R}^{n,n}$ is diagonal. Then $\mathcal{M}(T^*;\mathcal{B}) = \mathcal{M}(T;\mathcal{B})^\dagger = \mathcal{M}(T;\mathcal{B})$, and so we must have $T^* = T$.
- (1) $\implies$ (2): Proof by induction on $\dim V$. For the base case, let $\dim V = 1$. Since T is guaranteed to have one eigenvalue, it has an eigenvector v and necessarily $\operatorname{span}(v) = V$.

  Now suppose that $\dim V = n > 1$ and that (1) $\implies$ (2) for all vector spaces U with $\dim U \leq n-1$ and all self-adjoint $S \in \mathcal{L}(U)$. Let $T \in \mathcal{L}(V)$ be self-adjoint. Let $u \in V$ be an eigenvector of T with $\|u\| = 1$, and let $U = \operatorname{span}(u)$. Then U is a 1-dimensional subspace of V that is invariant under T. Thus $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

  By the induction hypothesis, there exists an ONB $\mathcal{B}_\perp = u_1, \ldots, u_{n-1}$ of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. But then $\mathcal{B} = u_1, \ldots, u_{n-1}, u$ is an ONB of V consisting of eigenvectors of T.

$\square$

End of Lecture 30