BEGINNING OF LECTURE 31

7.C Positive Operators and Isometries

Positive Operators

Definition 51. An operator $T \in \mathcal{L}(V)$ is \underline{positive} if $T$ is self-adjoint and

$$\forall v \in V, \quad \langle Tv, v \rangle \geq 0.$$ 

Examples:

1. Orthogonal projections $P_U$ (when $U$ is a subspace of $V$)

2. $T$ self-adjoint and $b, c \in \mathbb{R}$ such that $b^2 < 4c$, then $T^2 + bT + cI$ is a positive operator (see our proof proving that $T^2 + bT + cI$ is invertible)

Definition 52. An operator $R$ is the \underline{square root} of an operator $T$ if $R^2 = T$.

Example: Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is a rotation by the angle $\theta \in [0, 2\pi)$, i.e.,

$$T = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

If $R$ is a rotation by $\theta/2$,

$$R = \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

then $R^2 = T$.

Positive operators mimic the numbers $[0, \infty)$. The next two theorems formalize this statement.

Theorem 30. Let $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. $T$ is positive

2. $T$ is self-adjoint and all eigenvalues of $T$ are nonnegative

3. $T$ has a positive square root

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4. \( T \) has a self-adjoint square root

5. There exists an operator \( R \in \mathcal{L}(V) \) such that \( T = R^*R \)

**Proof.** The plan is: (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (1).

- (1) \( \Rightarrow \) (2): By definition \( T \) is self-adjoint. So let \( \lambda \) be an eigenvalue of \( T \) with eigenvector \( v \) (recall this means \( v \neq 0 \)). Then:
  \[
  0 \geq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2 \Rightarrow \lambda \geq 0
  \]

- (2) \( \Rightarrow \) (3): Since \( T \) is self-adjoint, by The Spectral Theorem there is an ONB \( e_1, \ldots, e_n \) of \( V \) consisting of eigenvectors of \( T \); let \( \lambda_1, \ldots, \lambda_n \) be the corresponding eigenvalues. By assumption each \( \lambda_k \geq 0 \). Define \( R \in \mathcal{L}(V) \) by defining it on \( e_1, \ldots, e_n \):
  \[
  Re_k = \sqrt{\lambda_k}e_k
  \]
  We claim that \( R \) is a positive operator and that \( R^2 = T \). The second point is clear since:
  \[
  R^2e_k = \lambda_k e_k = Te_k, \forall k = 1, \ldots, n
  \]
  Thus \( R^2 \) and \( T \) agree on a basis and so they must be the same operator.

  Furthermore \( R \) is positive since:
  \[
  \langle Rv, v \rangle = \left\langle R \left( \sum_{j=1}^{n} \langle v, e_j \rangle e_j \right), \sum_{k=1}^{n} \langle v, e_k \rangle e_k \right\rangle = \left\langle \sum_{j=1}^{n} \langle v, e_j \rangle Re_j, \sum_{k=1}^{n} \langle v, e_k \rangle e_k \right\rangle
  \]
  \[
  = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle \langle v, e_j \rangle Re_j, \langle v, e_k \rangle e_k \rangle
  \]
  \[
  = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle v, e_j \rangle \overline{\langle v, e_k \rangle} \langle Re_j, e_k \rangle
  \]
  \[
  = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle v, e_j \rangle \overline{\langle v, e_k \rangle} \sqrt{\lambda_j} e_j, e_k
  \]
  \[
  = \sum_{j=1}^{n} \sum_{k=1}^{n} \langle v, e_j \rangle \overline{\langle v, e_k \rangle} \sqrt{\lambda_j} \langle e_j, e_k \rangle
  \]
  \[
  = \sum_{j=1}^{n} \sqrt{\lambda_j} \cdot |\langle v, e_j \rangle|^2 \geq 0
  \]
• (3) ⇒ (4): By definition

• (4) ⇒ (5): (4) means that $T = R^2$ and $R = R^*$. Thus: $T = R^2 = RR = R^*R$.

• (5) ⇒ (1): We need to show $T$ is self-adjoint and $\langle Tv, v \rangle \geq 0$ for all $v \in V$. For the first part,

$$T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T$$

For the second part,

$$\langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle = \|Rv\|^2 \geq 0, \ \forall v \in V$$

\[ \square \]

**Theorem 31.** Every positive operator has a unique positive square root.

*Proof.* Suppose $T \in \mathcal{L}(V)$ is positive. Since $T$ is self-adjoint, by The Spectral Theorem it has an ONB $B$ of eigenvectors. Let $v \in B$ be one of these eigenvectors, and let $\lambda$ be its associated eigenvalue so that $Tv = \lambda v$. By the previous theorem $\lambda \geq 0$ and $T$ has a positive square root, say $R$. We will prove that $Rv = \sqrt{\lambda}v$. Thus $R$ will be uniquely determined on the basis $B$, which means that it is the unique positive square root of $T$.

Now we prove that $Rv = \sqrt{\lambda}v$. Since $R$ is positive, and hence self-adjoint, The Spectral Theorem implies that there exists an ONB $e_1, \ldots, e_n$ of $V$ consisting of eigenvectors of $R$. Let $\eta_1, \ldots, \eta_n$ be the corresponding eigenvalues; because $R$ is also positive, we know from the previous theorem that $\eta_k \geq 0$ for all $k$. Define $\lambda_k = \eta_k^2$; then $\sqrt{\lambda_k} = \eta_k$ and

$$Re_k = \sqrt{\lambda_k}e_k$$

Since $e_1, \ldots, e_n$ is an ONB, we can write

$$v = \sum_{k=1}^{n} \langle v, e_k \rangle e_k$$

Thus:

$$Rv = \sum_{k=1}^{n} \langle v, e_k \rangle \sqrt{\lambda_k} e_k \implies R^2v = \sum_{k=1}^{n} \langle v, e_k \rangle \lambda_k e_k$$
But $R^2 = T$ and $Tv = \lambda v$, so $R^2v = Tv = \lambda v$ which implies:

$$\sum_{k=1}^{n} \langle v, e_k \rangle \lambda_k e_k = \sum_{k=1}^{n} \langle v, e_k \rangle \lambda e_k \implies \sum_{k=1}^{n} \langle v, e_k \rangle (\lambda - \lambda_k) e_k = 0$$

$$\implies \langle v, e_k \rangle (\lambda - \lambda_k) = 0, \quad \forall k = 1, \ldots, n$$

Hence either $\langle v, e_k \rangle = 0$ or $\lambda = \lambda_k$ for each $k$; thus:

$$v = \sum_{\{k : \lambda_k = \lambda\}} \langle v, e_k \rangle e_k \implies Rv = \sum_{\{k : \lambda_k = \lambda\}} \langle v, e_k \rangle \sqrt{\lambda_k} e_k$$

$$= \sqrt{\lambda} \sum_{\{k : \lambda_k = \lambda\}} \langle v, e_k \rangle e_k = \sqrt{\lambda} v$$

\[ \square \]

End of Lecture 31