Beginning of Lecture 31

7.C Positive Operators and Isometries

Positive Operators

Definition 51. An operator $T \in \mathcal{L}(V)$ is positive if T is self-adjoint and

$$\forall\, v \in V,\ \ \langle Tv, v\rangle \geq 0.$$

Examples:

1. Orthogonal projections P_U (when U is a subspace of V)
2. T self-adjoint and $b, c \in \mathbb{R}$ such that $b^2 < 4c$, then $T^2 + bT + cI$ is a positive operator (see our proof proving that $T^2 + bT + cI$ is invertible)

Definition 52. An operator R is the square root of an operator T if $R^2 = T$.

Example: Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is a rotation by the angle $\theta \in [0, 2\pi)$, i.e.,

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

If R is a rotation by $\theta/2$,

$$R = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

then $R^2 = T$.

Positive operators mimic the numbers $[0, \infty)$. The next two theorems formalize this statement.

Theorem 30. *Let $T \in \mathcal{L}(V)$. Then the following are equivalent:*

1. *T is positive*
2. *T is self-adjoint and all eigenvalues of T are nonnegative*
3. *T has a positive square root*

4. *T has a self-adjoint square root*

5. *There exists and operator $R \in \mathcal{L}(V)$ such that $T = R^*R$*

Proof. The plan is: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

- (1) $\Rightarrow$ (2): By definition T is self-adjoint. So let λ be an eigenvalue of T with eigenvector v (recall this means $v \neq 0$). Then:

$$0 \geq \langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle = \lambda\|v\|^2 \Rightarrow \lambda \geq 0$$

- (2) $\Rightarrow$ (3): Since T is self-adjoint, by The Spectral Theorem there is an ONB $e_1, \ldots, e_n$ of V consisting of eigenvectors of T; let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. By assumption each $\lambda_k \geq 0$. Define $R \in \mathcal{L}(V)$ be defining it on $e_1, \ldots, e_n$:

$$Re_k = \sqrt{\lambda_k}e_k$$

We claim that R is a positive operator and that $R^2 = T$. The second point is clear since:

$$R^2e_k = \lambda_k e_k = Te_k, \forall\, k = 1, \ldots, n$$

Thus R^2 and T agree on a basis and so they must be the same operator. Furthermore R is positive since:

$$\begin{aligned}
\langle Rv, v\rangle &= \left\langle R\left(\sum_{j=1}^n \langle v, e_j\rangle e_j\right), \sum_{k=1}^n \langle v, e_k\rangle e_k\right\rangle = \left\langle \sum_{j=1}^n \langle v, e_j\rangle Re_j, \sum_{k=1}^n \langle v, e_k\rangle e_k\right\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \langle\langle v, e_j\rangle Re_j, \langle v, e_k\rangle e_k\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \langle v, e_j\rangle \overline{\langle v, e_k\rangle} \langle Re_j, e_k\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \langle v, e_j\rangle \overline{\langle v, e_k\rangle} \langle \sqrt{\lambda_j}e_j, e_k\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^n \langle v, e_j\rangle \overline{\langle v, e_k\rangle} \sqrt{\lambda_j}\langle e_j, e_k\rangle \\
&= \sum_{j=1}^n \sqrt{\lambda_j} \cdot |\langle v, e_j\rangle|^2 \geq 0
\end{aligned}$$

- (3) $\Rightarrow$ (4): By definition
- (4) $\Rightarrow$ (5): (4) means that $T = R^2$ and $R = R^*$. Thus: $T = R^2 = RR = R^*R$.
- (5) $\Rightarrow$ (1): We need to show T is self-adjoint and $\langle Tv, v\rangle \geq 0$ for all $v \in V$. For the first part,

$$T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T$$

For the second part,

$$\langle Tv, v\rangle = \langle R^*Rv, v\rangle = \langle Rv, Rv\rangle = \|Rv\|^2 \geq 0, \ \ \forall\, v \in V$$

□

Theorem 31. *Every positive operator has a unique positive square root.*

Proof. Suppose $T \in \mathcal{L}(V)$ is positive. Since T is self-adjoint, by The Spectral Theorem it has an ONB $\mathcal{B}$ of eigenvectors. Let $v \in \mathcal{B}$ be one of these eigenvectors, and let λ be its associated eigenvalue so that $Tv = \lambda v$. By the previous theorem $\lambda \geq 0$ and T has a positive square root, say R. We will prove that $Rv = \sqrt{\lambda}v$. Thus R will be uniquely determined on the basis $\mathcal{B}$, which means that it is the unique positive square root of T.

Now we prove that $Rv = \sqrt{\lambda}v$. Since R is positive, and hence self-adjoint, The Spectral Theorem impies that there exists an ONB $e_1, \ldots, e_n$ of V consisting of eigenvectors of R. Let $\eta_1, \ldots, \eta_n$ be the corresponding eigenvalues; because R is also positive, we know from the previous theorem that $\eta_k \geq 0$ for all k. Define $\lambda_k = \eta_k^2$; then $\sqrt{\lambda_k} = \eta_k$ and

$$Re_k = \sqrt{\lambda_k} e_k$$

Since $e_1, \ldots, e_n$ is an ONB, we can write

$$v = \sum_{k=1}^{n} \langle v, e_k\rangle e_k$$

Thus:

$$Rv = \sum_{k=1}^{n} \langle v, e_k\rangle \sqrt{\lambda_k}\, e_k \Longrightarrow R^2 v = \sum_{k=1}^{n} \langle v, e_k\rangle \lambda_k e_k$$

But $R^2 = T$ and $Tv = \lambda v$, so $R^2 v = Tv = \lambda v$ which implies:

$$\begin{aligned}\sum_{k=1}^{n} \langle v, e_k \rangle \lambda_k e_k = \sum_{k=1}^{n} \langle v, e_k \rangle \lambda e_k &\Longrightarrow \sum_{k=1}^{n} \langle v, e_k \rangle (\lambda - \lambda_k) e_k = 0 \\ &\Longrightarrow \langle v, e_k \rangle (\lambda - \lambda_k) = 0, \ \ \forall\, k = 1, \ldots, n\end{aligned}$$

Hence either $\langle v, e_k \rangle = 0$ or $\lambda = \lambda_k$ for each k; thus:

$$\begin{aligned}v = \sum_{\{k\,:\,\lambda_k = \lambda\}} \langle v, e_k \rangle e_k \Longrightarrow Rv &= \sum_{\{k\,:\,\lambda_k = \lambda\}} \langle v, e_k \rangle \sqrt{\lambda_k} e_k \\ &= \sqrt{\lambda} \sum_{\{k\,:\,\lambda_k = \lambda\}} \langle v, e_k \rangle e_k = \sqrt{\lambda} v\end{aligned}$$

□

End of Lecture 31