Beginning of Lecture 32

Isometries

Definition 53. An operator $S \in \mathcal{L}(V)$ is an isometry if

$$\forall\, v \in V, \quad \|Sv\| = \|v\|$$

Thus an isometry is an operator that preserves norms, or equivalently, preserves distances since the definition implies:

$$\forall\, u, w \in V, \quad \|S(u-w)\| = \|u-w\|$$

Example: Suppose $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ with $|\lambda_k| = 1$ for each k. Let $e_1, \ldots, e_n \in V$ be an ONB, and let $S \in \mathcal{L}(V)$ satisfy:

$$Se_k = \lambda_k e_k, \quad \forall\, k$$

Then we can show that S is an isometry.

Proof. Let $v \in V$. Then:

$$v = \sum_{k=1}^{n} \langle v, e_k \rangle e_k$$

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2$$

Thus:

$$\begin{aligned} Sv &= \sum_{k=1}^{n} \langle v, e_k \rangle S e_k \\ &= \sum_{k=1}^{n} \lambda_k \langle v, e_k \rangle e_k \\ \Rightarrow \|Sv\|^2 &= \sum_{k=1}^{n} |\lambda_k|^2 |\langle v, e_k \rangle|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 = \|v\|^2 \end{aligned}$$

□

Theorem 32. *Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent:*

1. *S is an isometry*
2. $\langle Su, Sv\rangle = \langle u, v\rangle, \quad \forall\, u, v \in V$
3. *If $e_1, \dots, e_m$ is orthonormal in V, then $Se_1, \dots, Se_m$ is orthonormal.*
4. *There exists an ONB $e_1, \dots, e_n$ such that $Se_1, \dots, Se_n$ is orthonormal*
5. $S^*S = I$
6. $SS^* = I$
7. *S^* is an isometry*
8. *S is invertible and $S^{-1} = S^*$*

Proof. Proof in parts:

- (1) $\implies$ (2): If $\mathbb{F} = \mathbb{R}$ we use the formula:

  $$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)$$

  while if $\mathbb{F} = \mathbb{C}$ we use:

  $$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i)$$

  For example, if $\mathbb{F} = \mathbb{R}$ then:

  $$\begin{aligned}\langle Su, Sv\rangle &= \frac{1}{4}(\|Su+Sv\|^2 - \|Su-Sv\|^2)\\ &= \frac{1}{4}(\|S(u+v)\|^2 - \|S(u-v)\|^2)\\ &= \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2)\\ &= \langle u, v\rangle\end{aligned}$$

  $\mathbb{F} = \mathbb{C}$ is similar and you should verify it on your own.

- (2) $\implies$ (3): This one is easy since:

  $$\langle Se_j, Se_k\rangle = \langle e_j, e_k\rangle = \delta(j-k)$$

- (3) $\Longrightarrow$ (4): Obvious
- (4) $\Longrightarrow$ (5): First note we have:

$$\langle e_j, e_k\rangle = \delta(j-k) = \langle Se_j, Se_k\rangle = \langle S^*Se_j, e_k\rangle, \quad \forall\, j,k$$

Now:

$$\begin{aligned}
\langle S^*Su, v\rangle &= \left\langle S^*S\sum_{j=1}^n a_je_j, \sum_{k=1}^n b_ke_k\right\rangle \\
&= \left\langle \sum_{j=1}^n a_jS^*Se_j, \sum_{k=1}^n b_ke_k\right\rangle \\
&= \sum_{j=1}^n\sum_{k=1}^n \langle a_jS^*Se_j, b_ke_k\rangle \\
&= \sum_{j=1}^n\sum_{k=1}^n a_j\bar{b}_k\langle S^*Se_j, e_k\rangle \\
&= \sum_{j=1}^n\sum_{k=1}^n a_j\bar{b}_k\langle e_j, e_k\rangle \\
&= \langle u, v\rangle \quad \text{[just unwind what we did to get to previous line]}
\end{aligned}$$

- (5) $\Longrightarrow$ (6): Since I is invertible, S^*S is invertible, which by 3.D #9 (HW 3) means that S^* and S are invertible. So we have:

$$S^*S = I \Rightarrow S^*SS^{-1} = S^{-1} \Rightarrow SS^* = SS^{-1} = I$$

- (6) $\Longrightarrow$ (7): We compute:

$$\|S^*v\|^2 = \langle S^*v, S^*v\rangle = \langle SS^*v, v\rangle = \langle v, v\rangle = \|v\|^2$$

- (7) $\Longrightarrow$ (8): We have proven already:

$$S \text{ an isometry } \Longrightarrow S^*S = I \text{ and } SS^* = I$$

Replacing S with S^* and using $(S^*)^* = S$ we obtain:

$$S^* \text{ an isometry } \Longrightarrow SS^* = I \text{ and } S^*S = I$$

Therefore, by definition S is invertible with $S^{-1} = S^*$.

- (8) $\implies$ (1): $S^{-1} = S^*$ implies $S^*S = I$. Thus:

$$\|Sv\|^2 = \langle Sv, Sv \rangle = \langle S^*Sv, v \rangle = \langle v, v \rangle = \|v\|^2$$

□

Theorem 33 (Spectral Theorem for Isometries when $\mathbb{F} = \mathbb{C}$)**.** *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $S \in \mathcal{L}(V)$*. Then the following are equivalent:*

1. *S is an isometry*

2. *There is an ONB of V consisting of eigenvectors of S whose corresponding eigenvalues all have complex modulus equal to* 1*.*

Proof. The example earlier proved (2) $\implies$ (1). So we just have to prove (1) $\implies$ (2). Since S is an isometry, the previous theorem implies $S^*S = I = SS^*$ which means that S is normal. Therefore we can apply the Complex Spectral Theorem. Thus there exists an ONB $e_1, \ldots, e_n$ consisting of eigenvectors of S; let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then:

$$|\lambda_k| = |\lambda_k|\|e_k\| = \|\lambda_k e_k\| = \|Se_k\| = \|e_k\| = 1$$

□

End of Lecture 32