Beginning of Lecture 33

Rigid Motions in $\mathbb{R}^n$

Definition 54. A rigid motion in an inner product space V is a transformation $f : V \to V$ that preserves distances, i.e.,

$$\forall\, u, v \in V, \quad \|f(u) - f(v)\| = \|u - v\|$$

Note, f is not assumed to be linear!

Examples:

- Any isometry $S \in \mathcal{L}(V)$ is a rigid motion.
- The translation map is a rigid motion, i.e., fix $w \in V$. Then the map:

  $$f(v) = v + w$$

  is a rigid motion, since

  $$\|f(u) - f(v)\| = \|u + w - (v + w)\| = \|u + w - v - w\| = \|u - v\|$$

  Note that translations are not linear unless $w = 0$. Indeed, $f(0) = w$, and we know that all linear maps take 0 to 0.

We are now going to prove that *every* rigid motion on a *real* inner product space is the composition of an isometry and a translation.

Theorem 34. *Let $f : V \to V$ be a rigid motion on a real inner product space V, and let $S(v) = f(v) - f(0)$. Then S is an isometry.*

Note the above theorem implies that $f(v) = S(v) + f(0)$, i.e., f can be decomposed as an isometry plus a translation.

To prove the theorem, we need the following lemma:

Lemma 3. *Let f be a rigid motion on a real inner product space V, and let $S(v) = f(v) - f(0)$. Then:*

1. $\|S(v)\| = \|v\|, \ \forall\, v \in V$
2. $\|S(u) - S(v)\| = \|u - v\|, \ \forall\, u, v \in V$

3. $\langle S(u), S(v)\rangle = \langle u, v\rangle, \ \forall\, u, v \in V$

Proof. We prove each statement individually:

1. Notice that:

$$\|S(v)\| = \|f(v) - f(0)\| = \|v - 0\| = \|v\|$$

2. A similar calculation yields:

$$\begin{aligned} \|S(u) - S(v)\| &= \|(f(u) - f(0)) - (f(v) - f(0))\| \\ &= \|f(u) - f(0) - f(v) + f(0)\| \\ &= \|f(u) - f(v)\| \\ &= \|u - v\| \end{aligned}$$

3. Using part 1 and the fact that we are working in a real inner product space we have:

$$\begin{aligned} \|S(u) - S(v)\|^2 &= \langle S(u) - S(v), S(u) - S(v)\rangle \\ &= \langle S(u), S(u)\rangle - \langle S(u), S(v)\rangle - \langle S(v), S(u)\rangle + \langle S(v), S(v)\rangle \\ &= \|S(u)\|^2 + \|S(v)\|^2 - 2\langle S(u), S(v)\rangle \\ &= \|u\|^2 + \|v\|^2 - 2\langle S(u), S(v)\rangle \end{aligned}$$

On the other hand, using part 2 we also have:

$$\begin{aligned} \|S(u) - S(v)\|^2 &= \|u - v\|^2 \\ &= \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle \end{aligned}$$

Therefore,

$$\|u\|^2 + \|v\|^2 - 2\langle S(u), S(v)\rangle = \|u\|^2 + \|v\|^2 - 2\langle u, v\rangle \Rightarrow \langle S(u), S(v)\rangle = \langle u, v\rangle$$

□

Now we can prove our theorem:

Proof of Theorem 34. By Lemma 3, part 1 we know that $\|S(v)\| = \|v\|$ for all $v \in V$. So S preserves the norm, but now we need to show that S is linear.

Let $e_1, \ldots, e_n$ be an ONB of V, and define:

$$\forall\, k = 1, \ldots, n, \quad g_k = S(e_k)$$

Note that $g_1, \ldots, g_n$ is in fact an ONB too. Indeed, using Lemma 3, part 3,

$$\langle g_j, g_k \rangle = \langle S(e_j), S(e_k) \rangle = \langle e_j, e_k \rangle = \delta(j-k)$$

Let $v \in V$ and write v as a linear combination of the ONB $e_1, \ldots, e_n$:

$$v = \sum_{k=1}^{n} a_k e_k, \quad a_k = \langle v, e_k \rangle$$

Do the same for $S(v)$ in the ONB $g_1, \ldots, g_n$:

$$S(v) = \sum_{k=1}^{n} b_k g_k, \quad b_k = \langle S(v), g_k \rangle$$

Using Lemma 3, part 3 again:

$$b_k = \langle S(v), g_k \rangle = \langle S(v), S(e_k) \rangle = \langle v, e_k \rangle = a_k$$

Therefore,

$$S\left(\sum_{k=1}^{n} a_k e_k\right) = \sum_{k=1}^{n} a_k g_k = \sum_{k=1}^{n} a_k S(e_k)$$

Thus S is a linear map (recall the proof of 3.5 in the book). $\square$

This concludes the material covered on the second midterm. In particular the second midterm will cover Chapter 6, sections 7.A, 7.B, 7.C from Chapter 7, and this material on rigid motions.

End of Lecture 33