Math 414: Linear Algebra II, Fall 2015
Final Exam

December 17, 2015

<u>NAME:</u>

This is a **closed book single author exam**. Use of books, notes, or other aids is *not* permissible, nor is collaboration with any of your fellow students.

You must **prove, justify, or explain** all of your assertions.

This final exam is out of 200 **points**.

Please write your **name** above, and at the **top** of each subsequent page.

Question 1:

True or False. NO explanation required.

(a) [5 points] Every linear operator has at least one eigenvalue.

Solution: False.

(b) [5 points] All sublists of a linearly dependent list of vectors are linearly dependent.

Solution: False.

(c) [5 points] The sum of two isometries is an isometry.

Solution: False.

(d) [5 points] A subspace of a finite dimensional vector space is finite dimensional.

Solution: True.

(e) [5 points] $\mathcal{P}_8(\mathbb{F})$ is isomorphic to $\mathbb{F}^{3,3}$.

Solution: True.

Question 2:

(a) [25 points] Suppose V and W are finite dimensional vector spaces over a field $\mathbb{F}$. Fix a vector $v_0 \in V$. Define $E \subset \mathcal{L}(V, W)$ as:

$$E = \{T \in \mathcal{L}(V, W) : Tv_0 = 0\}.$$

Prove E is a subspace of $\mathcal{L}(V, W)$.

Solution: We need to show three things:

(a) $0 \in E$: Clearly $0v_0 = 0$, so $0 \in E$.

(b) Closed under vector addition: Let $S, T \in E$. Then $(S + T)v_0 = Sv_0 + Tv_0 = 0 + 0 = 0$, and so $S + T \in E$.

(c) Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ and let $T \in E$. Then $(\lambda T)v_0 = \lambda Tv_0 = \lambda \cdot 0 = 0$, and so $\lambda T \in E$.

QUESTION 2:

(b) [25 points] Suppose V and W are finite dimensional vector spaces over a field $\mathbb{F}$. Fix a vector $v_0 \in V$. Define $E \subset \mathcal{L}(V, W)$ as:

$$E = \{T \in \mathcal{L}(V, W) : Tv_0 = 0\}.$$

Suppose $v_0 \neq 0$. Prove:

$$\dim E = (\dim V)(\dim W) - \dim W.$$

Hint: Consider the linear map $F : \mathcal{L}(V, W) \to W$ defined as:

$$F(T) = Tv_0, \quad \forall\, T \in \mathcal{L}(V, W).$$

What is $\operatorname{null} F$?

Solution: Define a linear map $F : \mathcal{L}(V, W) \to W$ as:

$$F(T) = Tv_0.$$

Notice that $\operatorname{null} F = E$. Furthermore, for any $w \in W$ there exists a linear map $T \in \mathcal{L}(V, W)$ such that $Tv_0 = w$. Therefore F is surjective. Then by the Rank-Nullity Theorem:

$$\begin{aligned} \dim \mathcal{L}(V, W) &= \dim \operatorname{range} F + \dim \operatorname{null} F, \\ (\dim V)(\dim W) &= \dim W + \dim E. \end{aligned}$$

QUESTION 3:

(a) [25 points] Let V be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V, \mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V, \mathbb{R})$ be linear functionals on V. Define $L : V \times V \to \mathbb{R}$ as

$$L(u, v) = \varphi_1(u)\varphi_2(v).$$

Prove that L is a bilinear form on V.

Solution: Note that φ_1 and φ_2 are linear, and let $\alpha, \beta \in \mathbb{R}$ and $u, v, w \in V$. Then:

$$\begin{aligned} L(\alpha u + \beta v, w) &= \varphi_1(\alpha u + \beta v)\varphi_2(w), \\ &= (\alpha\varphi_1(u) + \beta\varphi_1(v))\varphi_2(w), \\ &= \alpha\varphi_1(u)\varphi_2(w) + \beta\varphi_1(v)\varphi_2(w), \\ &= \alpha L(u, w) + \beta L(v, w). \end{aligned}$$

The proof for the second slot is identical.

QUESTION 3:

(b) [25 points] Let V be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V, \mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V, \mathbb{R})$ be linear functionals on V. Define $L : V \times V \to \mathbb{R}$ as

$$L(u, v) = \varphi_1(u)\varphi_2(v).$$

Recall from class that every bilinear form on a real inner product space can be written as $\langle Tu, v\rangle$, where $T \in \mathcal{L}(V)$ uniquely determines the bilinear form. Prove that there exists a unique $w \in V$, depending only on φ_1, and a unique $x \in V$, depending only on φ_2, such that

$$L(u, v) = \langle Tu, v\rangle, \quad \text{with } Tu = \langle u, w\rangle x.$$

Solution: By the Riesz Representation Theorem there exists a unique $x \in V$ such that:

$$\varphi_2(v) = \langle v, x\rangle, \quad \forall\, v \in V.$$

Therefore, using the above and the fact that V is a real inner product space,

$$L(u, v) = \varphi_1(u)\varphi_2(v) = \varphi_1(u)\langle v, x\rangle = \varphi_1(u)\langle x, v\rangle = \langle \varphi_1(u)x, v\rangle.$$

Thus, $L(u, v) = Tu$, where T can be written as $Tu = \varphi_1(u)x$. But now use the Riesz Representation Theorem again to assert there exists a unique $w \in V$ such that

$$\varphi_1(u) = \langle u, w\rangle, \quad \forall\, u \in V.$$

Then:

$$Tu = \varphi_1(u)x = \langle u, w\rangle x.$$

QUESTION 3:

(c) [25 points] Let V be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V,\mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V,\mathbb{R})$ be linear functionals on V. Define $L : V \times V \to \mathbb{R}$ as

$$L(u,v) = \varphi_1(u)\varphi_2(v).$$

Define a quadratic form $Q : V \to \mathbb{R}$ as:

$$Q(v) = L(v,v) = \varphi_1(v)\varphi_2(v).$$

Recall from class that every quadratic form on a real inner product space can be uniquely represented as $\langle Tv, v\rangle$ where $T \in \mathcal{L}(V)$ is self-adjoint. Using the vectors $w, x \in V$ from part (b), prove

$$Tv = \frac{1}{2}\Big(\langle v,w\rangle x + \langle v,x\rangle w\Big) \quad \text{is self-adjoint}$$

and

$$Q(v) = \langle Tv, v\rangle.$$

Solution: From part (b) we know that:

$$Q(v) = \langle Sv, v\rangle, \quad Sv = \langle v,w\rangle x.$$

You may recall from the book or the homework (or you can easily recompute on the exam) that:

$$S^*v = \langle v,x\rangle w.$$

Thus $T = \frac{1}{2}(S+S^*)$ is easily seen to be self-adjoint and furthermore (see notes from class on both points):

$$Q(v) = \langle Sv, v\rangle = \Big\langle \frac{1}{2}(S+S^*)v, v\Big\rangle.$$

Thus we have proven both facts about:

$$Tv = \frac{1}{2}(S+S^*)v = \frac{1}{2}\Big(\langle v,w\rangle x + \langle v,x\rangle w\Big).$$

QUESTION 4:

(a) [25 points] Suppose V and W are finite dimensional inner product spaces over a field $\mathbb{F}$, and suppose $T \in \mathcal{L}(V, W)$. Let $\hat{\sigma}$ denote the smallest singular value of T (possibly equal to zero), and let σ denote the largest singular value of T. Prove that:

$$\hat{\sigma}\|v\| \leq \|Tv\| \leq \sigma\|v\|, \quad \forall\, v \in V.$$

Solution: Let $\sigma_1, \ldots, \sigma_n$ be the singular values of T, with $n = \dim V$, and let $\sigma_1, \ldots, \sigma_r$, $r \leq n$, be the nonzero singular values of T. The SVD of T can be written:

$$Tv = \sum_{k=1}^{r} \sigma_k \langle v, e_k \rangle f_k,$$

where $e_1, \ldots, e_r \in V$ are orthonormal and $f_1, \ldots, f_r \in W$ are orthonormal. Note the we can extend $e_1, \ldots, e_r$ to an ONB of V, $e_1, \ldots, e_n$, and that

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2.$$

Thus:

$$\|Tv\|^2 = \sum_{k=1}^{r} \sigma_k^2 |\langle v, e_k \rangle|^2 \leq \sum_{k=1}^{r} \sigma^2 |\langle v, e_k \rangle|^2 \leq \sigma^2 \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 = \sigma^2 \|v\|^2,$$

which implies $\|Tv\| \leq \sigma\|v\|$. For the lower bound, if $\hat{\sigma} = 0$, then obviously $\|Tv\| \geq 0$. If $\hat{\sigma} > 0$, then $r = n$ and:

$$\|Tv\|^2 = \sum_{k=1}^{n} \sigma_k^2 |\langle v, e_k \rangle|^2 \geq \sum_{k=1}^{n} \hat{\sigma}^2 |\langle v, e_k \rangle|^2 = \hat{\sigma}^2 \|v\|^2,$$

which implies $\|Tv\| \geq \hat{\sigma}\|v\|$.

QUESTION 4:

(b) [25 points] Suppose V and W are finite dimensional inner product spaces over a field $\mathbb{F}$, and suppose $S, T \in \mathcal{L}(V, W)$. Define $R = S + T \in \mathcal{L}(V, W)$. Let r denote the largest singular value of R, let s denote the largest singular value of S, and let t denote the largest singular value of T. Prove:

$$r \leq s + t.$$

Solution: Recall $|R| = \sqrt{R^*R}$ and that r is an eigenvalue of $|R|$; let $v \in V$ be a corresponding eigenvector. Then:

$$\begin{aligned} r\|v\| &= \|rv\|, \\ &= \||R|v\|, \\ &= \|Rv\|, \\ &= \|Sv + Tv\|, \\ &\leq \|Sv\| + \|Tv\|, \\ &\leq s\|v\| + t\|v\|. \end{aligned}$$

Dividing both sides by $\|v\|$ yields $r \leq s + t$.

An alternate solution is to recall that the operator norm equals the largest singular value of the linear map, and that it is indeed a norm that satisfies the triangle inequality. Then:

$$r = \|R\| = \|S + T\| \leq \|S\| + \|T\| = s + t.$$