This is a closed book single author exam. Use of books, notes, or other aids is not permissible, nor is collaboration with any of your fellow students.

You must prove, justify, or explain all of your assertions.

This final exam is out of 200 points.

Please write your name above, and at the top of each subsequent page.
Question 1:

True or False. NO explanation required.

(a) [5 points] Every linear operator has at least one eigenvalue.
   Solution: False.

(b) [5 points] All sublists of a linearly dependent list of vectors are linearly dependent.
   Solution: False.

(c) [5 points] The sum of two isometries is an isometry.
   Solution: False.

(d) [5 points] A subspace of a finite dimensional vector space is finite dimensional.
   Solution: True.

(e) [5 points] $P_8(\mathbb{F})$ is isomorphic to $\mathbb{F}^{3,3}$.
   Solution: True.
Question 2:

(a) [25 points] Suppose $V$ and $W$ are finite dimensional vector spaces over a field $\mathbb{F}$. Fix a vector $v_0 \in V$. Define $E \subset \mathcal{L}(V,W)$ as:

$$E = \{ T \in \mathcal{L}(V,W) : Tv_0 = 0 \}.$$

Prove $E$ is a subspace of $\mathcal{L}(V,W)$.

Solution: We need to show three things:

(a) $0 \in E$: Clearly $0v_0 = 0$, so $0 \in E$.

(b) Closed under vector addition: Let $S, T \in E$. Then $(S + T)v_0 = Sv_0 + Tv_0 = 0 + 0 = 0$, and so $S + T \in E$.

(c) Closed under scalar multiplication: Let $\lambda \in \mathbb{F}$ and let $T \in E$. Then $(\lambda T)v_0 = \lambda Tv_0 = \lambda \cdot 0 = 0$, and so $\lambda T \in E$. 


Question 2:

(b) [25 points] Suppose $V$ and $W$ are finite dimensional vector spaces over a field $F$. Fix a vector $v_0 \in V$. Define $E \subset \mathcal{L}(V, W)$ as:

$$E = \{ T \in \mathcal{L}(V, W) : Tv_0 = 0 \}.$$

Suppose $v_0 \neq 0$. Prove:

$$\dim E = (\dim V)(\dim W) - \dim W.$$

**Hint:** Consider the linear map $F : \mathcal{L}(V, W) \to W$ defined as:

$$F(T) = Tv_0, \quad \forall T \in \mathcal{L}(V, W).$$

What is null $F$?

**Solution:** Define a linear map $F : \mathcal{L}(V, W) \to W$ as:

$$F(T) = Tv_0.$$

Notice that null $F = E$. Furthermore, for any $w \in W$ there exists a linear map $T \in \mathcal{L}(V, W)$ such that $Tv_0 = w$. Therefore $F$ is surjective. Then by the Rank-Nullity Theorem:

$$\dim \mathcal{L}(V, W) = \dim \text{range } F + \dim \text{null } F,$$

$$(\dim V)(\dim W) = \dim W + \dim E.$$
Question 3:

(a) [25 points] Let $V$ be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V, \mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V, \mathbb{R})$ be linear functionals on $V$. Define $L : V \times V \to \mathbb{R}$ as

$$L(u, v) = \varphi_1(u)\varphi_2(v).$$

Prove that $L$ is a bilinear form on $V$.

Solution: Note that $\varphi_1$ and $\varphi_2$ are linear, and let $\alpha, \beta \in \mathbb{R}$ and $u, v, w \in V$. Then:

$$L(\alpha u + \beta v, w) = \varphi_1(\alpha u + \beta v)\varphi_2(w),$$

$$= (\alpha \varphi_1(u) + \beta \varphi_1(v))\varphi_2(w),$$

$$= \alpha \varphi_1(u)\varphi_2(w) + \beta \varphi_1(v)\varphi_2(w),$$

$$= \alpha L(u, w) + \beta L(v, w).$$

The proof for the second slot is identical.
Question 3:

(b) [25 points] Let $V$ be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V, \mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V, \mathbb{R})$ be linear functionals on $V$. Define $L : V \times V \to \mathbb{R}$ as

$$L(u, v) = \varphi_1(u) \varphi_2(v).$$

Recall from class that every bilinear form on a real inner product space can be written as $\langle Tu, v \rangle$, where $T \in \mathcal{L}(V)$ uniquely determines the bilinear form. Prove that there exists a unique $w \in V$, depending only on $\varphi_1$, and a unique $x \in V$, depending only on $\varphi_2$, such that

$$L(u, v) = \langle Tu, v \rangle, \quad \text{with } Tu = \langle u, w \rangle x.$$ 

Solution: By the Riesz Representation Theorem there exists a unique $x \in V$ such that:

$$\varphi_2(v) = \langle v, x \rangle, \quad \forall v \in V.$$ 

Therefore, using the above and the fact that $V$ is a real inner product space,

$$L(u, v) = \varphi_1(u) \varphi_2(v) = \varphi_1(u) \langle v, x \rangle = \varphi_1(u) \langle x, v \rangle = \langle \varphi_1(u)x, v \rangle.$$ 

Thus, $L(u, v) = Tu$, where $T$ can be written as $Tu = \varphi_1(u)x$. But now use the Riesz Representation Theorem again to assert there exists a unique $w \in V$ such that

$$\varphi_1(u) = \langle u, w \rangle, \quad \forall u \in V.$$ 

Then:

$$Tu = \varphi_1(u)x = \langle u, w \rangle x.$$
**Question 3:**

(c) [25 points] Let $V$ be a finite dimensional inner product space over $\mathbb{R}$, and let $\varphi_1 \in \mathcal{L}(V, \mathbb{R})$ and $\varphi_2 \in \mathcal{L}(V, \mathbb{R})$ be linear functionals on $V$. Define $L : V \times V \rightarrow \mathbb{R}$ as

$$L(u, v) = \varphi_1(u)\varphi_2(v).$$

Define a quadratic form $Q : V \rightarrow \mathbb{R}$ as:

$$Q(v) = L(v, v) = \varphi_1(v)\varphi_2(v).$$

Recall from class that every quadratic form on a real inner product space can be uniquely represented as $\langle Tv, v \rangle$ where $T \in \mathcal{L}(V)$ is self-adjoint. Using the vectors $w, x \in V$ from part (b), prove

$$Tv = \frac{1}{2} \left( \langle v, w \rangle x + \langle v, x \rangle w \right)$$

is self-adjoint

and

$$Q(v) = \langle Tv, v \rangle.$$

**Solution:** From part (b) we know that:

$$Q(v) = \langle Sv, v \rangle, \quad Sv = \langle v, w \rangle x.$$

You may recall from the book or the homework (or you can easily recompute on the exam) that:

$$S^*v = \langle v, x \rangle w.$$

Thus $T = \frac{1}{2}(S + S^*)$ is easily seen to be self-adjoint and furthermore (see notes from class on both points):

$$Q(v) = \langle Sv, v \rangle = \left\langle \frac{1}{2}(S + S^*)v, v \right\rangle.$$

Thus we have proven both facts about:

$$Tv = \frac{1}{2}(S + S^*)v = \frac{1}{2} \left( \langle v, w \rangle x + \langle v, x \rangle w \right).$$
**Question 4:**

(a) [25 points] Suppose $V$ and $W$ are finite dimensional inner product spaces over a field $\mathbb{F}$, and suppose $T \in \mathcal{L}(V, W)$. Let $\hat{\sigma}$ denote the smallest singular value of $T$ (possibly equal to zero), and let $\sigma$ denote the largest singular value of $T$. Prove that:

$$\hat{\sigma} \|v\| \leq \|Tv\| \leq \sigma \|v\|, \quad \forall v \in V.$$ 

**Solution:** Let $\sigma_1, \ldots, \sigma_n$ be the singular values of $T$, with $n = \dim V$, and let $\sigma_1, \ldots, \sigma_r$, $r \leq n$, be the nonzero singular values of $T$. The SVD of $T$ can be written:

$$Tv = \sum_{k=1}^{r} \sigma_k \langle v, e_k \rangle f_k,$$

where $e_1, \ldots, e_r \in V$ are orthonormal and $f_1, \ldots, f_r \in W$ are orthonormal. Note the we can extend $e_1, \ldots, e_r$ to an ONB of $V$, $e_1, \ldots, e_n$, and that

$$\|v\|^2 = \sum_{k=1}^{n} |\langle v, e_k \rangle|^2.$$

Thus:

$$\|Tv\|^2 = \sum_{k=1}^{r} \sigma_k^2 |\langle v, e_k \rangle|^2 \leq \sum_{k=1}^{r} \sigma^2 |\langle v, e_k \rangle|^2 \leq \sigma^2 \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 = \sigma^2 \|v\|^2,$$

which implies $\|Tv\| \leq \sigma \|v\|$. For the lower bound, if $\hat{\sigma} = 0$, then obviously $\|Tv\| \geq 0$. If $\hat{\sigma} > 0$, then $r = n$ and:

$$\|Tv\|^2 = \sum_{k=1}^{n} \sigma_k^2 |\langle v, e_k \rangle|^2 \geq \sum_{k=1}^{n} \hat{\sigma}^2 |\langle v, e_k \rangle|^2 = \hat{\sigma}^2 \|v\|^2,$$

which implies $\|Tv\| \geq \hat{\sigma} \|v\|$.

(b) [25 points] Suppose $V$ and $W$ are finite dimensional inner product spaces over a field $\mathbb{F}$, and suppose $S, T \in \mathcal{L}(V, W)$. Define $R = S + T \in \mathcal{L}(V, W)$. Let $r$ denote the largest singular value of $R$, let $s$ denote the largest singular value of $S$, and let $t$ denote the largest singular value of $T$. Prove:

$$r \leq s + t.$$ 

**Solution:** Recall $|R| = \sqrt{R^*R}$ and that $r$ is an eigenvalue of $|R|$; let $v \in V$ be a corresponding eigenvector. Then:

$$r\|v\| = \|rv\|,$$

$$= \| |R|v\|,$$

$$= \| Rv\|,$$

$$= \| Sv + Tv\|,$$

$$\leq \| Sv\| + \|Tv\|,$$

$$\leq s\|v\| + t\|v\|.$$ 

Dividing both sides by $\|v\|$ yields $r \leq s + t$.

An alternate solution is to recall that the operator norm equals the largest singular value of the linear map, and that it is indeed a norm that satisfies the triangle inequality. Then:

$$r = \|R\| = \|S + T\| \leq \|S\| + \|T\| = s + t.$$