BEGINNING OF LECTURE 34

7.D Polar Decomposition and Singular Value Decomposition

Polar Decomposition

First we recall the polar form of a complex number $z \in \mathbb{C}$. Let $z = x + iy$. Every complex number z can also be written in polar form as:

$$z = re^{i\theta}, \quad r \geq 0, \ \theta \in [0, 2\pi),$$

where

$$\begin{aligned} r &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Note that in the polar formulation, r is nonnegative (and positive if $z \neq 0$), and $|e^{i\theta}| = 1$. We are going prove a <u>polar decomposition</u> for operators $T \in \mathcal{L}(V)$, using the analogy:

$$\begin{aligned} r \geq 0 &\longleftrightarrow \text{ positive operators} \\ e^{i\theta} &\longleftrightarrow \text{ isometries} \end{aligned}$$

First, recall that $R \in \mathcal{L}(V)$ is a square root of $T \in \mathcal{L}(V)$ if $R^2 = T$, and if T is a positive operator then T has a <u>unique</u> positive square root.

<u>Notation:</u> If T is a positive operator, let $\sqrt{T}$ denote the unique positive square root of T.

Second, recall from HW9, 7.C, #4, that for any $T \in \mathcal{L}(V, W)$, $T^*T \in \mathcal{L}(V)$ and $TT^* \in \mathcal{L}(W)$ are positive operators, and thus have unique positive square roots.

Theorem 35 (Polar Decomposition Theorem). *If $T \in \mathcal{L}(V)$, then there exists an isometry $S \in \mathcal{L}(V)$ such that*

$$T = S\sqrt{T^*T}$$

Proof. Define a function $S_1 : \operatorname{range}\sqrt{T^*T} \to \operatorname{range} T$ as:

$$S_1(\sqrt{T^*T}v) = Tv$$

An outline of the proof is:

1. Show S_1 is well defined and linear
2. Extend S_1 to an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

To show S_1 is well defined, first we show:

$$\|Tv\| = \|\sqrt{T^*T}v\|, \quad \forall\, v \in V \tag{18}$$

Indeed,

$$\begin{aligned}
\|Tv\|^2 &= \langle Tv, Tv\rangle \\
&= \langle T^*Tv, v\rangle \\
&= \langle \sqrt{T^*T}\sqrt{T^*T}v, v\rangle \\
&= \langle \sqrt{T^*T}v, \sqrt{T^*T}v\rangle \ \text{[recall positive operators are self-adjoint]} \\
&= \|\sqrt{T^*T}v\|^2
\end{aligned}$$

Now to show S_1 is well defined, suppose that $\sqrt{T^*T}v_1 = \sqrt{T^*T}v_2$; we must show that $Tv_1 = Tv_2$. We have:

$$\begin{aligned}
\|Tv_1 - Tv_2\| &= \|T(v_1 - v_2)\| \\
&= \|\sqrt{T^*T}(v_1 - v_2)\| \ \text{[by (18)]} \\
&= \|\sqrt{T^*T}v_1 - \sqrt{T^*T}v_2\| \\
&= 0
\end{aligned}$$

You should verify on your own that S_1 is linear. That completes part 1.

Now we extend $S_1 \in \mathcal{L}(\operatorname{range}\sqrt{T^*T}, \operatorname{range} T)$ so an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$. First note that by (18) and the definition of S_1, we have:

$$\|S_1 u\| = \|u\|, \quad \forall\, u \in \operatorname{range}\sqrt{T^*T}$$

Note this implies that S_1 is injective since only 0 maps to 0. Thus:

$$\dim\operatorname{range}\sqrt{T^*T} = \dim\operatorname{range} T \Rightarrow \dim(\operatorname{range}\sqrt{T^*T})^\perp = \dim(\operatorname{range} T)^\perp$$

Let $e_1, \ldots, e_m$ be an ONB for $(\operatorname{range}\sqrt{T^*T})^\perp$ and let $f_1, \ldots, f_m$ be an ONB for $(\operatorname{range} T)^\perp$. Note, both ONBs have the same length! Define a second linear map $S_2 : (\operatorname{range}\sqrt{T^*T})^\perp \to (\operatorname{range} T)^\perp$ by

$$S_2\left(\underbrace{\sum_{k=1}^m a_k e_k}_{w}\right) = \underbrace{\sum_{k=1}^m a_k f_k}_{S_2 w}$$

Since $e_1, \ldots, e_m$ and $f_1, \ldots, f_m$ are ONBs, by the definition of S_2 it is clear that

$$\forall\, w \in (\operatorname{range}\sqrt{T^*T})^\perp, \quad \|S_2 w\| = \|w\|$$

Since:

$$V = \operatorname{range}\sqrt{T^*T} \oplus (\operatorname{range}\sqrt{T^*T})^\perp$$

we have for each $v \in V$,

$$v = u + w, \quad u \in \operatorname{range}\sqrt{T^*T}, \;\; w \in (\operatorname{range}\sqrt{T^*T})^\perp \;\; [u, w \text{ are unique}]$$

Thus we can define $S \in \mathcal{L}(V)$ as:

$$S(v) = S(u + w) = S_1 u + S_2 w$$

Then for each $v \in V$,

$$S\sqrt{T^*T}v = S(\sqrt{T^*T}v) = S(\sqrt{T^*T}v + 0) = S_1(\sqrt{T^*T}v) + S_2(0) = Tv + 0 = Tv$$

and so $T = S\sqrt{T^*T}$.

Finally, we need to show that S is an isometry, i.e., it preserves norms:

$$\|Sv\|^2 = \|S_1 u + S_2 w\|^2 = \|S_1 u\|^2 + \|S_2 w\|^2 = \|u\|^2 + \|w\|^2 = \|v\|^2$$

□

Thus we can decompose *any* operator T into two very nice operators: an isometry and a positive operator. Furthermore, when $\mathbb{F} = \mathbb{C}$, our Spectral Theory tells us that there exists an ONB $\mathcal{B}_1$ such that $\mathcal{M}(S; \mathcal{B}_1)$ is diagonal and another ONB $\mathcal{B}_2$ such that $\mathcal{M}(\sqrt{T^*T}; \mathcal{B}_2)$ is diagonal. Unfortunately in general $\mathcal{B}_1 \neq \mathcal{B}_2$!

End of Lecture 34