Beginning of Lecture 37

Singular Value Decomposition

Let V, W be finite dimensional inner product spaces over the field $\mathbb{F}$ with $\dim V = n$ and $\dim W = m$.

Definition 55. Suppose $T \in \mathcal{L}(V, W)$. The Hermitian square of T is $T^*T \in \mathcal{L}(V)$.

Proposition 62. *Suppose $T \in \mathcal{L}(V, W)$. Its Hermitian square $T^*T \in \mathcal{L}(V)$ is a positive operator.*

Proof. Need to show T^*T is self-adjoint and $\langle T^*Tv, v\rangle \geq 0$ for all $v \in V$.

- Self adjoint:
$$(T^*T)^* = T^*(T^*)^* = T^*T$$
- Nonnegative:
$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2 \geq 0$$

□

Recall that for any $T \in \mathcal{L}(V, W)$, $T^*T \in \mathcal{L}(V)$ is a positive operator, and thus has a unique positive squareroot $\sqrt{T^*T}$. We are going to call

$$|T| := \sqrt{T^*T}$$

the modulus of T. The modulus of T shows how "big" the operator T is:

Proposition 63. *For any $T \in \mathcal{L}(V, W)$,*

$$\||T|v\|_V = \|Tv\|_W, \quad \forall\, v \in V$$

Proof. For any $v \in V$,

$$\||T|v\|^2 = \langle |T|v, |T|v\rangle = \langle |T|^*|T|v, v\rangle = \langle |T|^2 v, v\rangle = \langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2$$

□

Recall our polar decomposition. In this language we have for any $T \in \mathcal{L}(V)$,

$$T = S\sqrt{T^*T} = S|T|, \quad S \text{ is an isometry}$$

We are now going to go down a different path with $|T|$.

Remark: Since T^*T is positive, the Spectral Theorem implies that V has an ONB $e_1, \ldots, e_n$ consisting of eigenvectors of T^*T with:

$$T^*Te_k = \lambda_k e_k, \quad \lambda_k \geq 0 \ \forall\, k$$

Then $e_1, \ldots, e_n$ are eigenvectors of $|T| = \sqrt{T^*T}$ with corresponding eigenvalues $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$.

Definition 56. Suppose $T \in \mathcal{L}(V, W)$. The singular values of T are the eigenvalues of $|T|$, i.e., if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T^*T, then $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ are the singular values of T.

Remark: Every $T \in \mathcal{L}(V, W)$ has $n = \dim V$ singular values.

Let $T \in \mathcal{L}(V, W)$ and let $\sigma_1, \ldots, \sigma_n$ be the singular values of T (counting multiplicities). Assume also that $\sigma_1, \ldots, \sigma_r$ are the non-zero singular values of T (also counting multiplicities), so that in particular $\sigma_k = 0$ for $k > r$.

By definition, the numbers $\sigma_1^2, \ldots, \sigma_n^2$ are eigenvalues of T^*T. Let $e_1, \ldots, e_n$ be an ONB of V consisting of eigenvectors of T^*T so that

$$T^*Te_k = \sigma_k^2 e_k, \quad \forall\, k = 1, \ldots, n \tag{19}$$

Proposition 64. *The system*

$$f_k := \frac{1}{\sigma_k} Te_k \in W, \quad k = 1, \ldots, r, \tag{20}$$

is an orthonormal system.

Proof. Observe:

$$\langle Te_j, Te_k \rangle = \langle T^*Te_j, e_k \rangle = \langle \sigma_k^2 e_j, e_k \rangle = \sigma_k^2 \langle e_j, e_k \rangle = \begin{cases} 0 & \text{if } j \neq k \\ \sigma_j^2 & \text{if } j = k \end{cases}$$

□

Theorem 36 (Singular Value Decomposition)**.** *Suppose $T \in \mathcal{L}(V, W)$ and let $\sigma_1, \ldots, \sigma_n$ be the singular values of T. Let $e_1, \ldots, e_n \in V$ be an ONB consisting of eigenvectors of T^*T satisfying* (19)*, and let $f_1, \ldots, f_r \in W$ be the orthonormal system defined by* (20)*. Then:*

$$Tv = \sum_{k=1}^{r} \sigma_k \langle v, e_k \rangle f_k, \quad \forall\, v \in V$$

Proof. Define $S \in \mathcal{L}(V, W)$ as:

$$Sv := \sum_{k=1}^{r} \sigma_k \langle v, e_k \rangle f_k$$

We are going to show $S = T$ by showing that S and T agree on a basis of V. In particular, take $e_1, \ldots, e_n$ which is an ONB for V. Then:

- For $j = 1, \ldots, r$,

$$Se_j = \sum_{k=1}^{k} \sigma_k \langle e_j, e_k \rangle f_k = \sigma_j \langle e_j, e_j \rangle f_j = \sigma_j \|e_j\|^2 f_j = \sigma_j f_j = Te_j$$

- For $j > r$,

$$Se_j = \sum_{k=1}^{r} \sigma_k \langle e_j, e_k \rangle f_k = 0 = Te_j$$

  where the last inequality $Te_j = 0$ follows since:

$$\|Te_j\| = \||T|e_j\| = \|\sigma_j e_j\| = 0$$

□

Recall that for a general linear map $T \in \mathcal{L}(V, W)$, we used one basis $\mathcal{B}_V = v_1, \ldots, v_n$ for V and another basis $\mathcal{B}_W = w_1, \ldots, w_m$ for W to construct the matrix $A = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)$ which has entries defined as:

$$Tv_k = \sum_{j=1}^{m} A_{j,k} w_j$$

Take $\mathcal{B}_V = e_1, \ldots, e_n$ (as above), and extend $f_1, \ldots, f_r$ to an ONB $\mathcal{B}_W = f_1, \ldots, f_r, f_{r+1}, \ldots, f_m$ of W. The Singular Value Decomposition (SVD) says that $A = \mathcal{M}(T; \mathcal{B}_V, \mathcal{B}_W)$ has a "diagonal structure", i.e.,

$$\begin{aligned} Te_k &= \sum_{j=1}^{m} A_{j,k} f_k \\ &= \sum_{j=1}^{r} \sigma_j \langle e_k, e_j \rangle f_j = \sigma_k f_k \text{ [since } \sigma_k = 0 \text{ for } k > r] \\ &\implies A_{j,k} = \begin{cases} \sigma_k & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \end{aligned}$$

When $V = W$ this means $\mathcal{M}(T; (e_1, \ldots, e_n), (f_1, \ldots, f_n))$ is diagonal!

End of Lecture 37