Beginning of Lecture 39

Application 2: Operator norm of a linear transformation

Let $T \in \mathcal{L}(V, W)$ and recall that $B := \{v \in V : \|v\| \leq 1\}$ is the closed unit ball. Consider the following optimization problem:

$$\max_{v \in B} \|Tv\|$$

Let's first consider a positive operator $T \in \mathcal{L}(V)$. Suppose $e_1, \ldots, e_n$ is an ONB for V consisting of eigenvectors of T with eigenvalues $\sigma_1, \ldots, \sigma_n \geq 0$. Note that since T is positive, T is self-adjoint, and so $|T| = \sqrt{T^*T} = \sqrt{T^2} = T$. Therefore the singular values of T are also $\sigma_1, \ldots, \sigma_n$. Let σ_1 the largest singular value. We are going to show that:

$$\sigma_1 = \max_{v \in B} \|Tv\| \tag{22}$$

Note that:

$$v = \sum_{k=1}^{n} \langle v, e_k \rangle e_k$$
$$\Rightarrow Tv = \sum_{k=1}^{n} \langle v, e_k \rangle T e_k = \sum_{k=1}^{n} \sigma_k \langle v, e_k \rangle e_k$$

Thus for any $v \in V$, and in particular $v \in B$,

$$\|Tv\|^2 = \sum_{k=1}^{n} \sigma_k^2 |\langle v, e_k \rangle|^2 \leq \sigma_1^2 \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 = \sigma_1^2 \|v\|^2$$
$$\Rightarrow \|Tv\|^2 \leq \sigma_1^2 \|v\|^2$$

On the other hand:

$$\|Te_1\| = \|\sigma_1 e_1\| = \sigma_1 \|e_1\|$$

Thus we have shown (22).

Now let $T \in \mathcal{L}(V, W)$ and let $\sigma_1, \ldots, \sigma_n$ be the singular values of T, with $\sigma_1, \ldots, \sigma_r$ nonzero and σ_1 the largest singular value. We will use the singular value decomposition of T:

$$Tv = \sum_{k=1}^{r} \sigma_k \langle v, e_k \rangle f_k$$

For any $v \in V$, and in particular $v \in B$,

$$\|Tv\|^2 = \sum_{k=1}^{r} \sigma_k^2 |\langle v, e_k \rangle|^2 \leq \sigma_1^2 \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 = \sigma_1^2 \|v\|^2$$

Additionally,

$$\|Te_1\| = \|\sigma_1 f_1\| = \sigma_1 \|f_1\| = \sigma_1 \|e_1\|$$

where the last inequality follows because $e_1, \ldots, e_n$ and $f_1, \ldots, f_r$ are both orthonormal, and hence $\|e_1\| = \|f_1\| = 1$. Therefore in the general case as well we see that

$$\sigma_1 = \max_{v \in B} \|Tv\| \tag{23}$$

Definition 57. The quantity

$$\|T\| := \max\{\|Tv\| : v \in V,\ \|v\| \leq 1\}$$

is the operator norm of T.

It is easy to see that $\|T\|$ is indeed a norm on $\mathcal{L}(V, W)$, meaning that:

- $\|\lambda T\| = |\lambda| \|T\|$ for all $\lambda \in \mathbb{F}$
- $\|T + S\| \leq \|T\| + \|S\|$ for all $S, T \in \mathcal{L}(V, W)$
- $\|T\| \geq 0$ for all $T \in \mathcal{L}(V, W)$
- $\|T\| = 0 \iff T = 0$

One of the main properties of the operator norm is:

$$\|Tv\| \leq \|T\| \|v\|$$

We have shown if σ_1 is the largest singular value of T, then $\|T\| = \sigma_1$. One can also show $\|T\| = C_0$, where $C_0 \in \mathbb{R}$ is defined as:

$$C_0 = \min\{C \in \mathbb{R} : \|Tv\| \leq C\|v\|,\ \ \forall\, v \in V\}$$

Other equivalent formulations of the operator norm are:

$$\|T\| = \max\{\|Tv\| : v \in V,\ \|v\| = 1\} = \max\left\{\frac{\|Tv\|}{\|v\|} : v \in V,\ v \neq 0\right\}$$

End of Lecture 39