Beginning of Lecture 41

Example: Let's do an example to show the problems that can occur with an ill conditioned matrix. Consider the system of equations:

$$\left.\begin{array}{rcl} x_1 + x_2 &=& 2 \\ x_1 + 1.001x_2 &=& 2 \end{array}\right\} \Longleftrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1.001 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b}$$

The solution is:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = A^{-1}b$$

Now let's consider the same system but with a small perturbation

$$\Delta b = \begin{pmatrix} 0 \\ 0.001 \end{pmatrix}$$

With the perturbation, the system now is:

$$\left.\begin{array}{rcl} \tilde{x}_1 + \tilde{x}_2 &=& 2 \\ \tilde{x}_1 + 1.001\tilde{x}_2 &=& 2.001 \end{array}\right\} \Longleftrightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1.001 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}}_{\tilde{x}} = \underbrace{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}_{b} + \underbrace{\begin{pmatrix} 0 \\ 0.001 \end{pmatrix}}_{\Delta b}$$

The singular values of A are approximately $\sigma_1 \approx 2.0005$ and $\sigma_2 \approx 0.0005$. Thus the condition number of A is approximately (!):

$$\|A\|\|A^{-1}\| = \frac{\sigma_1}{\sigma_2} \approx \frac{2.0005}{0.0005} \approx 4000$$

The new solution is easily seen to be:

$$\tilde{x} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix}}_{x=A^{-1}b} + \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\Delta x = A^{-1}\Delta b}$$

which is completely different than x. Notice:

$$\text{Ratio of initial data perturbation} = \frac{\|\Delta b\|}{\|b\|} = \frac{\sqrt{8}}{0.001} \approx 0.00035$$

$$\text{Ratio of solution perturbation} = \frac{\|\Delta x\|}{\|x\|} = \frac{\sqrt{2}}{2} \approx 0.7$$

8 Bilinear and Quadratic Forms

Bilinear Forms

Definition 58. Let V and W be vector spaces over a field $\mathbb{F}$. The product $V \times W$ is defined as:

$$V \times W := \{(v, w) : v \in V, \, w \in W\}$$

Proposition 65. *Let V and W be vector spaces over a field $\mathbb{F}$. Then $V \times W$ is a vector space over $\mathbb{F}$ with vector addition and scalar multiplication defined as:*

- *Vector addition:* $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$
- *Scalar multiplication:* $\lambda(v, w) = (\lambda v, \lambda w)$

Definition 59. Let V be a vector space over a field $\mathbb{F}$. A bilinear form on V is a function $L : V \times V \to \mathbb{F}$ that is linear in both arguments:

$$L(\alpha u + \beta v, w) = \alpha L(u, w) + \beta L(v, w), \quad \forall \, u, v, w \in V, \;\; \forall \, \alpha, \beta \in \mathbb{F}$$
$$L(u, \alpha v + \beta w) = \alpha L(u, v) + \beta L(u, w), \quad \forall \, u, v, w \in V, \;\; \forall \, \alpha, \beta \in \mathbb{F}$$

Examples:

1. Let $\varphi_1, \varphi_2 \in \mathcal{L}(V, \mathbb{F})$ be linear functionals on V. Define $L : V \times V \to \mathbb{F}$ as:
$$L(u, v) = \varphi_1(u)\varphi_2(v)$$
Then L is a bilinear form (as you can verify).

2. Let V be an inner product space over $\mathbb{R}$ and let $T \in \mathcal{L}(V)$. Then:
$$L(u, v) = \langle Tu, v \rangle$$
is a bilinear form. In fact every bilinear form on a real inner product space is of this form.

Theorem 38. *Let V be an inner product space over $\mathbb{R}$, and let $L : V \times V \to \mathbb{R}$ be a bilinear form on V. Then there exists a unique $T \in \mathcal{L}(V)$ such that*

$$L(u, v) = \langle Tu, v \rangle$$

Proof. Let $\mathcal{B} = e_1, \ldots, e_n$ be an ONB for V. Then:

$$u = \sum_{j=1}^{n} a_j e_j \quad \text{and} \quad v = \sum_{k=1}^{n} b_k e_k$$

Then:

$$\begin{aligned} L(u, v) &= L\left(\sum_{j=1}^{n} a_j e_j, v\right) \\ &= \sum_{j=1}^{n} a_j L(e_j, v) \\ &= \sum_{j=1}^{n} a_j L\left(e_j, \sum_{k=1}^{n} b_k e_k\right) \\ &= \sum_{j=1}^{n} a_j b_k L(e_j, e_k) \end{aligned}$$

Define $A \in \mathbb{R}^{n,n}$ as:

$$A_{k,j} = L(e_j, e_k)$$

Since $\mathcal{M}(\cdot; \mathcal{B}) : \mathcal{L}(V) \to \mathbb{R}^{n,n}$ is an isomorphism, there exists a unique $T \in \mathcal{L}(V)$ such that

$$\mathcal{M}(T; \mathcal{B}) = A$$

Note in particular, this means that

$$Te_j = \sum_{k=1}^{n} A_{k,j} e_k$$

We then have:

$$
\begin{aligned}
\langle Tu, v\rangle &= \left\langle T\left(\sum_{j=1}^{n} a_j e_j\right), v\right\rangle \\
&= \sum_{j=1}^{n} a_j \langle Te_j, v\rangle \\
&= \sum_{j=1}^{n} a_j \Big\langle \sum_{k=1}^{n} A_{k,j} e_k, v\Big\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j A_{k,j} \langle e_k, v\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j A_{k,j} \Big\langle e_k, \sum_{l=1}^{n} b_l e_l\Big\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_j b_l A_{k,j} \langle e_k e_l\rangle \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j b_k A_{k,j} \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j b_k L(e_j, e_k) = L(u, v)
\end{aligned}
$$

□

End of Lecture 41