Beginning of Lecture 42

Quadratic forms

Definition 60. Let V be a real inner product space. A quadratic form $Q : V \to \mathbb{R}$ is the "diagonal" of a bilinear form $L : V \times V \to \mathbb{R}$, i.e.,

$$Q(v) = L(v, v), \quad \text{for some bilinear form } L$$

Note by function of the form:

$$Q(v) = \langle Tv, v \rangle, \quad T \in \mathcal{L}(V)$$

Note by our previous theorem,

$$Q(v) = \langle Tv, v \rangle, \quad \text{for some } T \in \mathcal{L}(V)$$

Unlike bilinear forms, quadratic forms are not uniquely determined by $T \in \mathcal{L}(V)$ (we'll give an example in a bit). However, if restrict ourselves to self-adjoint T, then T is unique.

Proposition 66. *Let V be a finite dimensional real inner product space, and suppose $Q : V \to \mathbb{R}$ is a quadratic form on V. Then there exists a unique self-adjoint $T \in \mathcal{L}(V)$ such that*

$$Q(v) = \langle Tv, v \rangle, \quad T = T^*$$

Proof. We know that every quadratic form can be represented as $Q(v) = \langle \widetilde{T}v, v \rangle$ for some $\widetilde{T} \in \mathcal{L}(V)$. Define $T \in \mathcal{L}(V)$ as:

$$Tv = \frac{1}{2}(\widetilde{T} + \widetilde{T}^*)$$

Note that T is self-adjoint; indeed:

$$T^* = \left(\frac{1}{2}(\widetilde{T} + \widetilde{T}^*) \right)^* = \frac{1}{2}(\widetilde{T}^* + \widetilde{T}) = T$$

Additionally,

$$
\begin{aligned}
\langle Tv, v\rangle &= \langle \frac{1}{2}(\widetilde{T} + \widetilde{T}^*)v, v\rangle \\
&= \frac{1}{2}\langle \widetilde{T}v + \widetilde{T}^*v, v\rangle \\
&= \frac{1}{2}\langle \widetilde{T}v, v\rangle + \frac{1}{2}\langle \widetilde{T}^*v, v\rangle \\
&= \frac{1}{2}\langle \widetilde{T}v, v\rangle + \frac{1}{2}\langle v, \widetilde{T}v\rangle \\
&= \frac{1}{2}\langle \widetilde{T}v, v\rangle + \frac{1}{2}\langle \widetilde{T}v, v\rangle \\
&= \langle \widetilde{T}v, v\rangle = Q(v)
\end{aligned}
$$

Thus there exists a self-adjoint $T \in \mathcal{L}(V)$ such that $Q(v) = \langle Tv, v\rangle$.

We now show that T is unique. Suppose $S \in \mathcal{L}(V)$ is another self-adjoint operator such that $Q(v) = \langle Sv, v\rangle$. Then:

$$
\begin{aligned}
\langle Tv, v\rangle &= \langle Sv, v\rangle, \quad && \forall\, v \in V \\
\Rightarrow \langle (T - S)v, v\rangle &= 0, && \forall\, v \in V
\end{aligned}
$$

But

$$(T - S)^* = T^* - S^* = T - S$$

so $T - S$ is self-adjoint. But by 7.16 in the book, this means $T - S = 0$, i.e., $T = S$. □

Quadratic Forms on $\mathbb{R}^n$

On $\mathbb{R}^n$, we can identify operators $T \in \mathcal{L}(\mathbb{R}^n)$ with their matrix $A = \mathcal{M}(T) \in \mathbb{R}^{n,n}$ in the standard basis. This means that quadratic forms $Q : \mathbb{R}^n \to \mathbb{R}$ can be written as:

$$Q(x) = \langle Ax, x\rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{j,k}x_jx_k, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Thus quadratic forms on $\mathbb{R}^n$ are homogeneous polynomials of degree 2, that is $Q \in \mathcal{P}_2(\mathbb{R}^n)$ and Q only contains terms of the form $a_{j,k}x_jx_k$.

Quadratic forms on $\mathbb{R}^n$ are uniquely represented by symmetric matrices, since $A = \mathcal{M}(T)$ is symmetric when $T = T^*$. There are though an infinite number of non-symmetric matrices that give the same quadratic form. For example, consider $Q : \mathbb{R}^2 \to \mathbb{R}$ defined as:

$$Q(x) = x_1^2 + x_2^2 - 4x_1x_2$$

Then

$$Q(x) = \langle Ax, x \rangle, \quad \text{for any } A = \begin{pmatrix} 1 & a-4 \\ -a & 1 \end{pmatrix}, \; a \in \mathbb{R}$$

Quadratic forms Q such that:

$$Q(x) = \sum_{k=1}^{n} a_k x_k^2$$

are particularly nice. Since $Q(x) = \langle Ax, x \rangle$, the nice form corresponds to A being a diagonal matrix wth $a_k = A_{k,k}$. In general though $A = \mathcal{M}(T)$ is not diagonal. Let $S \in \mathcal{L}(\mathbb{R}^n)$ be an isometry, and let $x = Sy$. Then:

$$Q(x) = Q(Sy) = \langle TSy, Sy \rangle = \langle S^*TSy, y \rangle$$

So we want an isometry S such that $\mathcal{M}(S^*TS)$ is diagonal. But since T is self-adjoint, by the Spectral Theorem there exists an ONB $e_1, \ldots, e_n$ of $\mathbb{R}^n$ consisting of eigenvectors of T. Define S so that:

$$\mathcal{M}(S) = \begin{pmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{pmatrix}$$

End of Lecture 42