Math 414: Linear Algebra II, Fall 2015
Midterm 1

October 12, 2015

Name:

This is a **closed book single author exam**. Use of books, notes, or other aids is *not* permissible, nor is collaboration with any of your fellow students.

You must **prove, justify, or explain** all of your assertions.

This midterm is out of 100 **points**.

[4 pts] Please write your **name** above, and at the **top** of each subsequent page.

Question 1:

True or False (explain each of your answers):

(a) [6 pts] If the vectors $v_1, \ldots, v_m$ span the vector space V, then every vector in V can be written as a linear combination of the vectors $v_1, \ldots, v_n$ in only one way.

(b) [6 pts] There exist vectors v_1, v_2, v_3 that are linearly dependent, but such that $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, and $w_3 = v_3 + v_1$ are linearly independent.

(c) [6 pts] If an operator has one eigenvector, it has infinitely many eigenvectors.

(d) [6 pts] The sum of two eigenvectors of an operator is always an eigenvector.

QUESTION 2:

A matrix $A \in \mathbb{F}^{n,n}$ is *symmetric* if $A_{j,k} = A_{k,j}$ for all $j, k = 1, \ldots, n$. Let $\text{Sym}(n)$ denote the set of all symmetric matrices, i.e.,

$$\text{Sym}(n) = \{A \in \mathbb{F}^{n,n} : A_{j,k} = A_{k,j} \ \ \forall\, j, k = 1, \ldots, n\}.$$

(a) [6 pts] Prove that $\text{Sym}(n)$ is a subspace of $\mathbb{F}^{n,n}$.

(b) [6 pts] Write down a basis for Sym(3).

(c) [6 pts] What is the dimension of Sym(3)? Explain your answer.

(d) [6 pts] Find a subspace U of $\mathbb{F}^{2,2}$ such that $\mathbb{F}^{2,2} = \text{Sym}(2) \oplus U$. For whatever U you write down, make sure your prove that $\mathbb{F}^{2,2} = \text{Sym}(2) \oplus U$.

QUESTION 3:

The set $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a 2-tuple $(x, y) \in \mathbb{R}^2$.

(a) [8 pts] Let $\alpha = a+ib \in \mathbb{C}$. Treating $\mathbb{C}$ as a *complex* vector space, show that the function

$$T : \mathbb{C} \to \mathbb{C}$$
$$T(z) = \alpha z$$

is a linear operator on $\mathbb{C}$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

(b) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as the *real* vector space $\mathbb{R}^2$, show that the function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x + iy) = \alpha(x + iy)$$

is a linear operator on $\mathbb{R}^2$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

(c) [8 pts] Define

$$T(x + iy) = 2x - y + i(x - 3y).$$

Show that this function is not a linear operator on the complex vector space $\mathbb{C}$, but it is a linear operator if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$.

Question 4:

Let V be a finite dimensional complex vector space. An operator $T \in \mathcal{L}(V)$ is called *nilpotent* if $T^k = 0$ for some $k \in \mathbb{Z}$, $k \geq 1$, i.e., $T^k v = 0$ for all $v \in V$.

(a) [12 pts] What are the eigenvalues of a nilpotent operator? Make sure to prove your assertion.

(b) [12 pts] For a nilpotent operator T, let $k_0 \in \mathbb{Z}$, $k_0 \geq 1$, be the *smallest* positive integer such that $T^{k_0} = 0$. So in particular:

$$\begin{aligned} &T^k \neq 0, \text{ for all } 1 \leq k < k_0, \\ &T^{k_0} = 0. \end{aligned}$$

For which k_0 is T diagonalizable and for which k_0 is T not diagonalizable? Make sure to prove your assertion.