This is a closed book single author exam. Use of books, notes, or other aids is not permissible, nor is collaboration with any of your fellow students.

You must prove, justify, or explain all of your assertions.

This midterm is out of 100 points.

[4 pts] Please write your name above, and at the top of each subsequent page.
Question 1:

True or False (explain each of your answers):

(a) [6 pts] If the vectors $v_1, \ldots, v_m$ span the vector space $V$, then every vector in $V$ can be written as a linear combination of the vectors $v_1, \ldots, v_n$ in only one way.

(b) [6 pts] There exist vectors $v_1, v_2, v_3$ that are linearly dependent, but such that $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, and $w_3 = v_3 + v_1$ are linearly independent.

(c) [6 pts] If an operator has one eigenvector, it has infinitely many eigenvectors.

(d) [6 pts] The sum of two eigenvectors of an operator is always an eigenvector.
A matrix $A \in \mathbb{F}^{n,n}$ is symmetric if $A_{j,k} = A_{k,j}$ for all $j, k = 1, \ldots, n$. Let $\text{Sym}(n)$ denote the set of all symmetric matrices, i.e.,

$$\text{Sym}(n) = \{A \in \mathbb{F}^{n,n} : A_{j,k} = A_{k,j} \ \forall \ j, k = 1, \ldots, n\}.$$

(a) [6 pts] Prove that $\text{Sym}(n)$ is a subspace of $\mathbb{F}^{n,n}$.

(b) [6 pts] Write down a basis for $\text{Sym}(3)$.

(c) [6 pts] What is the dimension of $\text{Sym}(3)$? Explain your answer.

(d) [6 pts] Find a subspace $U$ of $\mathbb{F}^{2,2}$ such that $\mathbb{F}^{2,2} = \text{Sym}(2) \oplus U$. For whatever $U$ you write down, make sure your prove that $\mathbb{F}^{2,2} = \text{Sym}(2) \oplus U$. 
Question 3:

The set $\mathbb{C}$ can be identified with $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a 2-tuple $(x, y) \in \mathbb{R}^2$.

(a) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as a complex vector space, show that the function

$$T : \mathbb{C} \to \mathbb{C}$$
$$T(z) = \alpha z$$

is a linear operator on $\mathbb{C}$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

(b) [8 pts] Let $\alpha = a + ib \in \mathbb{C}$. Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that the function

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$
$$T(x + iy) = \alpha(x + iy)$$

is a linear operator on $\mathbb{R}^2$. What is its matrix $\mathcal{M}(T)$ in the standard basis?

(c) [8 pts] Define

$$T(x + iy) = 2x - y + i(x - 3y).$$

Show that this function is not a linear operator on the complex vector space $\mathbb{C}$, but it is a linear operator if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$. 
Question 4:

Let $V$ be a finite dimensional complex vector space. An operator $T \in \mathcal{L}(V)$ is called nilpotent if $T^k = 0$ for some $k \in \mathbb{Z}$, $k \geq 1$, i.e., $T^k v = 0$ for all $v \in V$.

(a) [12 pts] What are the eigenvalues of a nilpotent operator? Make sure to prove your assertion.

(b) [12 pts] For a nilpotent operator $T$, let $k_0 \in \mathbb{Z}$, $k_0 \geq 1$, be the smallest positive integer such that $T^{k_0} = 0$. So in particular:

\[
T^k \neq 0, \text{ for all } 1 \leq k < k_0,
\]
\[
T^{k_0} = 0.
\]

For which $k_0$ is $T$ diagonalizable and for which $k_0$ is $T$ not diagonalizable? Make sure to prove your assertion.