This is a closed book single author exam. Use of books, notes, or other aids is not permissible, nor is collaboration with any of your fellow students.

You must prove, justify, or explain all of your assertions.

This midterm is out of 100 points.

Please write your name above, and at the top of each subsequent page.
**Question 1:**

Suppose $V$ is a finite dimensional inner product space and $T \in \mathcal{L}(V)$.

(a) [10 points] Let $v$ be an eigenvector of $T$ with eigenvalue $\lambda \in \mathbb{F}$. Suppose $|\lambda| < 1$. Prove that for all $\epsilon > 0$, there exists a positive integer $m$ such that $\|T^m v\| \leq \epsilon \|v\|$. 
(b) [20 points] Suppose $T$ is a self-adjoint operator such that if $\lambda \in \mathbb{F}$ is an eigenvalue of $T$, then $|\lambda| < 1$. Prove that for all $\epsilon > 0$, there exists a positive integer $m$ such that $\|T^m v\| \leq \epsilon \|v\|$ for all $v \in V$. 

**Question 2:**

Let $V, W$ be finite dimensional inner product spaces over the field $\mathbb{F}$. Let $T \in \mathcal{L}(V, W)$.

(a) [10 points] Prove that $T^*T \in \mathcal{L}(V)$ and $TT^* \in \mathcal{L}(W)$ are a positive operators.
(b) [20 points] Let $\epsilon > 0$ and let $I \in \mathcal{L}(V)$ be the identity operator on $V$. Prove that $T^*T + \epsilon I \in \mathcal{L}(V)$ is invertible.
(c) [20 points] Let $\lambda \in \mathbb{F}, \lambda \neq 0$. Show that $\lambda$ is eigenvalue of $T^*T$ if and only if $\lambda$ is an eigenvalue of $TT^*$. Furthermore, show that

$$\dim E(\lambda, T^*T) = \dim E(\lambda, TT^*)$$
**Question 3:**

[20 points] Let $\mathbb{C}^n$ be endowed with the standard inner product, i.e., for $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\langle w, z \rangle = \sum_{k=1}^{n} w_k \bar{z}_k$$

For each $n = 1, 2, \ldots$, give an example of a rigid motion $f : \mathbb{C}^n \to \mathbb{C}^n$ such that $f(0) = 0$ and $f$ is not linear.
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