Math 414: Linear Algebra II, Fall 2015
Midterm 2

November 23, 2015

Name:

This is a **closed book single author exam**. Use of books, notes, or other aids is *not* permissible, nor is collaboration with any of your fellow students.

You must **prove, justify, or explain** all of your assertions.

This midterm is out of 100 **points**.

Please write your **name** above, and at the **top** of each subsequent page.

QUESTION 1:

Suppose V is a finite dimensional inner product space and $T \in \mathcal{L}(V)$.

(a) [10 points] Let v be an eigenvector of T with eigenvalue $\lambda \in \mathbb{F}$. Suppose $|\lambda| < 1$. Prove that for all $\epsilon > 0$, there exists a positive integer m such that $\|T^m v\| \leq \epsilon \|v\|$.

Solution: First note:

$$Tv = \lambda v \Rightarrow T^m v = \lambda^m v \Rightarrow \|T^m v\| = |\lambda|^m \|v\|$$

So given that $|\lambda| < 1$, we need to show: For all $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that $|\lambda|^m \leq \epsilon$. But this is clear, since:

$$\begin{aligned} |\lambda|^m \leq \epsilon &\iff \log |\lambda|^m \leq \log \epsilon \\ &\iff m \log |\lambda| \leq \log \epsilon \\ &\iff m \geq \frac{\log \epsilon}{\log |\lambda|} \quad \text{[inequality switched b/c } |\lambda| < 1 \Rightarrow \log |\lambda| < 0] \end{aligned}$$

(b) [20 points] Suppose T is a self-adjoint operator such that if $\lambda \in \mathbb{F}$ is an eigenvalue of T, then $|\lambda| < 1$. Prove that for all $\epsilon > 0$, there exists a positive integer m such that $\|T^m v\| \leq \epsilon \|v\|$ for all $v \in V$.

Solution: Since T is self-adjoint, the Spectral Theorem implies that V has an ONB $e_1, \dots, e_n$ consisting of eigenvectors of T. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Thus:

$$\begin{aligned}
v &= \sum_{k=1}^{n} \langle v, e_k \rangle e_k \\
\Rightarrow T^m v &= \sum_{k=1}^{n} \lambda_k^m \langle v, e_k \rangle e_k \\
\Rightarrow \|T^m v\| &= \left\| \sum_{k=1}^{n} \lambda_k^m \langle v, e_k \rangle e_k \right\| \\
&\leq \sum_{k=1}^{n} |\lambda_k|^m |\langle v, e_k \rangle| \|e_k\| \quad \text{[Triangle Inequality]} \\
&\leq \sum_{k=1}^{n} |\lambda_k|^m \|v\| \|e_k\|^2 \quad \text{[Cauchy-Schwarz]} \\
&= \|v\| \sum_{k=1}^{n} |\lambda_k|^m \qquad (1)
\end{aligned}$$

Let

$$\delta = \max\{|\lambda_1|, \dots, |\lambda_n|\} < 1$$

Then using (1),

$$\begin{aligned}
\|T^m v\| &\leq \|v\| \sum_{k=1}^{n} |\lambda_k|^m \\
&\leq \|v\| \sum_{k=1}^{n} \delta^m \\
&\leq \|v\| n \delta^m
\end{aligned}$$

Thus we need to show, given that $\delta < 1$, there exist an $m \in \mathbb{N}$ such that $\delta^m < \epsilon/n$. But this is the same argument as in part (a).

QUESTION 2:

Let V, W be finite dimensional inner product spaces over the field $\mathbb{F}$. Let $T \in \mathcal{L}(V, W)$.

(a) [10 points] Prove that $T^*T \in \mathcal{L}(V)$ and $TT^* \in \mathcal{L}(W)$ are a positive operators.

Solution: We have:

$$\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle = \|Tv\|^2 \geq 0$$

and

$$\langle TT^*v, v\rangle = \langle T^*v, T^*v\rangle = \|T^*v\|^2 \geq 0$$

Additionally,

$$(T^*T)^* = T^*(T^*)^* = T^*T$$
$$(TT^*)^* = (T^*)^*T^* = TT^*$$

(b) [20 points] Let $\epsilon > 0$ and let $I \in \mathcal{L}(V)$ be the identity operator on V. Prove that $T^*T + \epsilon I \in \mathcal{L}(V)$ is invertible.

Solution: We will use the following from Chapter 5: An operator $S \in \mathcal{L}(V)$ with an upper triangular matrix $\mathcal{M}(S)$ is invertible if and only if all of the entries on the diagonal are nonzero.

Since T^*T is a positive operator, the Spectral Theorem tells us:

- All eigenvalues of T^*T are real valued and nonnegative.
- There exists an ONB $\mathcal{B} = e_1, \ldots e_n$ consisting of eigenvectors of T^*T such that $\mathcal{M}(T^*T; \mathcal{B})$ is diagonal.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T^*T; by the above remark, $\lambda_k \geq 0$ for each k and:

$$\mathcal{M}(T^*T; \mathcal{B}) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

We then have:

$$\mathcal{M}(T^*T + \epsilon I; \mathcal{B}) = \begin{pmatrix} \lambda_1 + \epsilon & & 0 \\ & \ddots & \\ 0 & & \lambda_n + \epsilon \end{pmatrix}$$

Thus $\mathcal{M}(T^*T + \epsilon I; \mathcal{B})$ is a diagonal matrix with all diagonal entries nonzero. By our remark from Chapter 5, this implies that $T^*T + \epsilon I$ is invertible.

(c) [20 points] Let $\lambda \in \mathbb{F}$, $\lambda \neq 0$. Show that λ is eigenvalue of T^*T if and only if λ is an eigenvalue of TT^*. Furthermore, show that

$$\dim E(\lambda, T^*T) = \dim E(\lambda, TT^*)$$

Solution: Suppose λ is an eigenvalue of T^*T with eigenvector $v \in V$, $v \neq 0$. Then $T^*Tv = \lambda v \neq 0$, which implies that $Tv \neq 0$. Consider then:

$$TT^*(Tv) = TT^*Tv = T\lambda v = \lambda Tv$$

Therefore Tv is an eigenvector of TT^* with eigenvalue λ. For the reverse implication, let $S = T^*$. Then we just showed that if λ is an eigenvalue of S^*S, then λ is an eigenvalue of SS^*. But $S^*S = TT^*$ and $SS^* = T^*T$.

Now let $e_1, \ldots, e_m$ be an ONB for $E(\lambda, T^*T)$. By the above, we have

$$\mathcal{L} = \frac{1}{\sqrt{\lambda}}Te_1, \ldots, \frac{1}{\sqrt{\lambda}}Te_m \in E(\lambda, TT^*).$$

We claim additionally that the list $\mathcal{L}$ is orthonormal. Indeed:

$$\begin{aligned}
\langle \frac{1}{\sqrt{\lambda}}Te_j, \frac{1}{\sqrt{\lambda}}Te_k \rangle &= \frac{1}{\lambda}\langle Te_j, Te_k \rangle \quad \text{[since } \lambda \in \mathbb{R}] \\
&= \frac{1}{\lambda}\langle e_j, T^*Te_k \rangle \\
&= \frac{1}{\lambda}\langle e_j, \lambda e_k \rangle \\
&= \langle e_j, e_k \rangle \\
&= \delta(j-k)
\end{aligned}$$

Thus $\mathcal{L}$ is orthonormal, which means it is linearly independent, which means that $\dim E(\lambda, TT^*) \geq m = \dim(\lambda, T^*T)$. For the other inequality, again let $S = T^*$. Then we just showed that $\dim E(\lambda, SS^*) \geq \dim(\lambda, S^*S)$. But since $SS^* = T^*T$ and $S^*S = TT^*$, we have $\dim E(\lambda, T^*T) \geq \dim E(\lambda, TT^*)$. Combining the two inequalities we have $\dim E(\lambda, T^*T) = \dim E(\lambda, TT^*)$.

Question 3:

[20 points] Let $\mathbb{C}^n$ be endowed with the standard inner product, i.e., for $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\langle w, z \rangle = \sum_{k=1}^{n} w_k \bar{z}_k$$

For each $n = 1, 2, \ldots$, give an example of a rigid motion $f : \mathbb{C}^n \to \mathbb{C}^n$ such that $f(0) = 0$ and f is not linear.

Solution: Take:

$$f(z) = f(z_1, \ldots, z_n) := (\bar{z}_1, \ldots, \bar{z}_n)$$

Then $f(0) = f(0, \ldots, 0) = (0, \ldots, 0) = 0$ but f is not linear since for $\lambda \in \mathbb{C}$,

$$f(\lambda z) = f(\lambda z_1, \ldots, \lambda z_n) = (\overline{\lambda z}_1, \ldots, \overline{\lambda z}_n) = (\bar{\lambda}\bar{z}_1, \ldots, \bar{\lambda}\bar{z}_n) = \bar{\lambda}(\bar{z}_1, \ldots, \bar{z}_n) = \bar{\lambda} f(z)$$