Uncertainty Principles in Sparse Representation and Compressed Sensing

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Outline

1. Uncertainty Principles
   - The Donoho-Stark Uncertainty Principle
   - An Uncertainty Principle for Cyclic Groups of Prime Order

2. Sparse Representation in Overcomplete Dictionaries
   - Time/Frequency Dictionary
   - Two Orthonormal Bases Dictionary
   - Any Overcomplete Spanning Set
   - Sparse Representation via Compressed Sensing

Outline

The Fourier Transform on $\mathbb{Z}/N\mathbb{Z}$

- $\ell^2(\mathbb{Z}/N\mathbb{Z}) := \{f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}\}$
- $\ell^2(\mathbb{Z}/N\mathbb{Z}) \cong \mathbb{C}^N$

Definition (Fourier Transform)

Let $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$. The *Fourier transform* of f, denoted $\hat{f}$, is defined as

$$\hat{f}(\omega) := \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} f(t) e^{-2\pi i \omega t/N}, \qquad \omega \in \mathbb{Z}/N\mathbb{Z}$$

The Donoho-Stark Uncertainty Principle

- $\text{supp}(f) := \{t \in \mathbb{Z}/N\mathbb{Z} \,:\, f(t) \neq 0\}$
- For any set A, let $|A|$ denote the cardinality of A.

Theorem (Donoho and Stark 1989)

If $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$ is a non-zero function, then

$$|\text{supp}(f)||\text{supp}(\hat{f})| \geq N \tag{1}$$

$$|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq 2\sqrt{N} \tag{2}$$

The Uncertainty Principle is Sharp

Example (The Picket Fence Signal)

- Let N be a perfect square.
- Set

$$\Pi(t) := \begin{cases} 1 & t = m\sqrt{N}, \quad m = 0, 1, \ldots, \sqrt{N} - 1 \\ 0 & \text{otherwise} \end{cases}$$

- $|\mathrm{supp}(\Pi)| = \sqrt{N}$
- $\hat{\Pi} = \Pi \Longrightarrow |\mathrm{supp}(\hat{\Pi})| = \sqrt{N}$
- Therefore we have equality in equations (1) and (2).
- Up to translation, modulation, and scalar multiplication, these are the only type of signal where equality is attained.

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Outline

The Uncertainty Principle for $\mathbb{Z}/p\mathbb{Z}$

- Let $N = p$, a prime number, and suppose $f \in \ell^2(\mathbb{Z}/p\mathbb{Z})$.
- Since p has no non-trivial factorizations, we would hope to improve upon the D-S uncertainty principle.

Theorem (Biró 1998; Meshulam 2005; Tao 2005)

If $f \in \ell^2(\mathbb{Z}/p\mathbb{Z})$ is a non-zero function, then

$$|\mathrm{supp}(f)| + |\mathrm{supp}(\hat{f})| \geq p + 1$$

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An Equivalent Theorem

- The uncertainty principle for $\mathbb{Z}/p\mathbb{Z}$ is in fact equivalent to the following much older theorem:

Theorem (Chebotarëv 1926)

Let p be a prime number and $1 \leq n \leq p$. Let $t_1, \ldots, t_n$ be distinct elements of $\mathbb{Z}/p\mathbb{Z}$ and let $\omega_1, \ldots, \omega_n$ also be distinct elements of $\mathbb{Z}/p\mathbb{Z}$. Then the matrix $(e^{2\pi i t_j \omega_k / p})_{1 \leq j,k \leq n}$ has non-zero determinant.

Outline

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Motivation

- Consider the following two orthonormal bases for $\ell^2(\mathbb{Z}/N\mathbb{Z})$:
  - Spikes: $\{\delta_\tau(t)\}_{\tau=0}^{N-1}$
  - Sinusoids: $\{\frac{1}{\sqrt{N}}e^{2\pi i\omega t/N}\}_{\omega=0}^{N-1}$
- Set $f(t) := \delta_0(t) + e^{2\pi i t/N}$.
- Spikes representation of f:

$$f(t) = \sum_{\tau=0}^{N-1} f(\tau)\delta_\tau(t) \quad \text{where} \quad f(\tau) = \begin{cases} 2, & \tau = 0 \\ e^{2\pi i\tau/N}, & \tau \neq 0 \end{cases}$$

- Sinusoids representation of f:

$$f(t) = \frac{1}{\sqrt{N}} \sum_{\omega=0}^{N-1} \hat{f}(\omega)e^{2\pi i\omega t/N} \quad \text{where} \quad \hat{f}(\omega) = \begin{cases} (N+1)/\sqrt{N}, & \omega = 1 \\ 1/\sqrt{N}, & \omega \neq 1 \end{cases}$$

Motivation

- Neither basis *individually* yields a sparse representation of f.
- Consider the *dictionary*

$$\Phi := \{\varphi_\gamma(t)\} = \{\varphi_{1,\tau}(t)\} \cup \{\varphi_{2,\omega}(t)\},$$

where

$$\varphi_{1,\tau}(t) = \delta_\tau(t) \quad \text{and} \quad \varphi_{2,\omega}(t) = \frac{1}{\sqrt{N}} e^{2\pi i \omega t/N}.$$

- $$f(t) = \sum_\gamma \alpha_\gamma \varphi_\gamma(t)$$

Motivation

- The sparsest representation of f in Φ will have two non-zero coefficients. One example is:

$$f(t) = \sum_{\gamma} \alpha_\gamma \varphi_\gamma(t) \quad \text{such that} \quad \alpha_\gamma = \begin{cases} 1, & \varphi_\gamma(t) = \delta_0(t) \\ \sqrt{N}, & \varphi_\gamma(t) = \frac{1}{\sqrt{N}} e^{2\pi i t/N} \\ 0, & \text{otherwise} \end{cases}$$

- We will be interested in the following:
  - When is a sparse representation *unique*?
  - How do we *find* sparse representations?

ℓ^0 Minimization

- Given any signal $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$, we would like to find the sparsest representation of f in the dictionary $\Phi = \{\text{spikes}\} \cup \{\text{sinusoids}\}$.
- $\alpha := (\alpha_\gamma) \in \mathbb{C}^{2N}$
- $\|\alpha\|_0 := |\{\gamma : \alpha_\gamma \neq 0\}|$
- We are trying to solve the following optimization problem:

Problem (ℓ^0 Minimization)

$$(P_0): \quad \arg\min \|\alpha\|_0 \quad \textit{subject to} \quad f = \Phi\alpha = \sum_{\gamma} \alpha_\gamma \varphi_\gamma$$

- We want the solution of P_0 to be unique.
- Solving P_0 is extremely difficult.

The Uniqueness of P_0

- Let T be a subset of the time domain, i.e., $T \subseteq \{(1, \tau)\}$.
- Let Ω be a subset of the frequency domain, i.e., $\Omega \subseteq \{(2, \omega)\}$.
- Suppose f has the following representation:

$$f = \Phi\alpha = \sum_{\gamma \in T} \alpha_\gamma \varphi_\gamma + \sum_{\gamma \in \Omega} \alpha_\gamma \varphi_\gamma$$

Theorem (Donoho and Huo 1999)

If

$$\|\alpha\|_0 = |T| + |\Omega| < \sqrt{N},$$

then α is the unique solution of P_0, i.e., the representation of f is unique and is the sparsest one possible.

Proof of P_0 Uniqueness

Proof.

- Let α^1 and α^2 be two representations of f, i.e.,

$$f = \Phi\alpha^1 = \Phi\alpha^2.$$

- Furthermore, assume $\|\alpha^1\|_0 < \sqrt{N}$ and $\|\alpha^2\|_0 < \sqrt{N}$.
- Set $\ker(\Phi) := \{\zeta \in \mathbb{C}^{2N} \,:\, \Phi\zeta = 0\} \Longrightarrow \alpha^1 - \alpha^2 \in \ker(\Phi)$.
- One can show: $\ker(\Phi) = \{(x, -\hat{x}) \,:\, x \in \mathbb{C}^N\}$
- $\alpha^1 - \alpha^2 = (x, -\hat{x}) \Longrightarrow \|\alpha^1 - \alpha^2\|_0 \geq 2\sqrt{N}$ or $\alpha^1 - \alpha^2 = 0$
- $\|\alpha^1\|_0 < \sqrt{N}$ and $\|\alpha^2\|_0 < \sqrt{N} \Longrightarrow \alpha^1 - \alpha^2 = 0$

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The Picket Fence Signal Revisted

Example (The Picket Fence Signal)

- Recall the picket fence signal:

$$\hat{\Pi}(t) = \Pi(t) = \begin{cases} 1 & t = m\sqrt{N}, \quad m = 0, 1, \ldots, \sqrt{N} - 1 \\ 0 & \text{otherwise} \end{cases}$$

- Therefore Π has two two distinct representations in Φ, each with $\sqrt{N}$ non-zero coefficients.

ℓ^1 Minimization

- Set $\|\alpha\|_1 := \sum_\gamma |\alpha_\gamma|$ and consider the following optimization problem:

Problem (Basis Pursuit)

$$(P_1): \quad \arg\min \|\alpha\|_1 \quad \textit{subject to} \quad f = \Phi\alpha = \sum_\gamma \alpha_\gamma \varphi_\gamma$$

- P_1 is a convex optimization problem and can be solved using linear programming.

P_0/P_1 Equivalence

- Again let $T \subseteq \{(1,\tau)\}$ and $\Omega \subseteq \{(2,\omega)\}$.
- Suppose f has the following representation:

$$f = \Phi\alpha = \sum_{\gamma\in T} \alpha_\gamma \varphi_\gamma + \sum_{\gamma\in\Omega} \alpha_\gamma \varphi_\gamma$$

Theorem (Donoho and Huo 1999)

If

$$\|\alpha\|_0 = |T| + |\Omega| < \frac{\sqrt{N}}{2},$$

then α is the unique solution of P_1, which is also the unique solution of P_0. Thus, the representation of f is unique, it is the sparsest one possible, and it may be found by solving the ℓ^1 minimization problem.

ℓ^1 Concentration Lemma

- Let $x \in \mathbb{C}^N$ with $x = (x_t)_{t=0}^{N-1}$ and $\hat{x} = (\hat{x}_\omega)_{\omega=0}^{N-1}$.
- Set

$$\mu(T,\Omega) := \sup_{x \neq 0} \frac{\sum_T |x_t| + \sum_\Omega |\hat{x}_\omega|}{\|x\|_1 + \|\hat{x}\|_1}.$$

Lemma (Donoho and Huo 1999)

If

$$\mu(T,\Omega) < \frac{1}{2},$$

then α is the unique solution to P_1.

Proof of Lemma

Proof.

- In order for α to be the unique solution of P_1, for every non-zero $\zeta \in \ker(\Phi)$, we must have:

$$\|\alpha + \zeta\|_1 - \|\alpha\|_1 > 0$$

- $$\|\alpha + \zeta\|_1 - \|\alpha\|_1 = \sum_{(T\cup\Omega)^c} |\zeta_\gamma| + \sum_{T\cup\Omega} (|\alpha_\gamma + \zeta_\gamma| - |\alpha_\gamma|)$$

- $|\alpha_\gamma + \zeta_\gamma| - |\alpha_\gamma| \geq -|\zeta_\gamma|$

- $$\|\alpha + \zeta\|_1 - \|\alpha\|_1 \geq \sum_{(T\cup\Omega)^c} |\zeta_\gamma| - \sum_{T\cup\Omega} |\zeta_\gamma|$$

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- $$\|\alpha + \zeta\|_1 - \|\alpha\|_1 \geq \sum_{(T\cup\Omega)^c} |\zeta_\gamma| - \sum_{T\cup\Omega} |\zeta_\gamma|$$

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Proof of Lemma

Proof.

- $$\|\alpha + \zeta\|_1 - \|\alpha\|_1 \geq \sum_{(T\cup\Omega)^c} |\zeta_\gamma| - \sum_{T\cup\Omega} |\zeta_\gamma|$$
- Hence a sufficient condition for uniqueness is:

$$\sum_{T\cup\Omega} |\zeta_\gamma| < \sum_{(T\cup\Omega)^c} |\zeta_\gamma| \qquad \forall\ \zeta \in \ker(\Phi) \quad (\zeta \neq 0) \tag{3}$$

- Recall $\zeta = (x, -\hat{x})$ for some $x \in \mathbb{C}^N$.
- Therefore, (3) is equivalent to:

$$\sum_T |x_t| + \sum_\Omega |\hat{x}_\omega| < \frac{1}{2}(\|x\|_1 + \|\hat{x}\|_1) \qquad \forall\ x \in \mathbb{C}^N \quad (x \neq 0)$$

□

Proof of Lemma

Proof.

- $$\|\alpha + \zeta\|_1 - \|\alpha\|_1 \geq \sum_{(T\cup\Omega)^c} |\zeta_\gamma| - \sum_{T\cup\Omega} |\zeta_\gamma|$$
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□

Bound on μ

- The following lemma will complete the proof:

Lemma (Donoho and Huo 1999)

$$\mu(T,\Omega) \leq \frac{|T| + |\Omega|}{\sqrt{N} + 1} \tag{4}$$

- Notice, if $|T| + |\Omega| < \sqrt{N}/2$, then (4) gives $\mu(T,\Omega) < 1/2$.

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Mutual Coherence

- Let Φ_1 and Φ_2 be orthonormal bases for $\ell^2(\mathbb{Z}/N\mathbb{Z})$.

Definition (Mutual Coherence)

The *mutual coherence* of Φ_1 and Φ_2 is defined as

$$M = M(\Phi_1, \Phi_2) := \sup\{|\langle \varphi_1, \varphi_2 \rangle| \ : \ \varphi_1 \in \Phi_1, \quad \varphi_2 \in \Phi_2\}$$

Examples

- $1/\sqrt{N} \leq M(\Phi_1, \Phi_2) \leq 1$
- $M(\text{spikes}, \text{sinusoids}) = 1/\sqrt{N}$
- If $\Phi_1 \cap \Phi_2 \neq \emptyset$, then $M(\Phi_1, \Phi_2) = 1$.

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Generalization of the D-S Uncertainty Principle

- A signal $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$ may be represented both as

$$f = \Phi_1 \beta^1 \quad \text{and} \quad f = \Phi_2 \beta^2$$

for some $\beta^1, \beta^2 \in \mathbb{C}^N$.

Theorem (Elad and Bruckstein 2002)

If $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$ is a non-zero function, then

$$\begin{aligned} \|\beta^1\|_0 \|\beta^2\|_0 &\geq 1/M^2 \\ \|\beta^1\|_0 + \|\beta^2\|_0 &\geq 2/M \end{aligned}$$

The Uniqueness of P_0

- Consider the dictionary

$$\Phi = \Phi_1 \cup \Phi_2.$$

- Suppose f has the following representation:

$$f = \Phi\alpha = \sum_{\gamma} \alpha_\gamma \varphi_\gamma$$

Theorem (Elad and Bruckstein 2002)

If

$$\|\alpha\|_0 < \frac{1}{M},$$

then α is the unique solution of P_0, i.e., the representation of f is unique and is the sparsest one possible.

P_0/P_1 Equivalence

Theorem (Elad and Bruckstein 2002)

If

$$\|\alpha\|_0 < \frac{(\sqrt{2}-0.5)}{M} \approx \frac{0.9142}{M},$$

then α is the unique solution of P_1, which is also the unique solution of P_0. Thus, the representation of f is unique, it is the sparsest one possible, and it may be found by solving the ℓ^1 minimization problem.

Theorem (Feuer and Nemirovski 2003)

The condition

$$\|\alpha\|_0 < \frac{(\sqrt{2}-0.5)}{M} \approx \frac{0.9142}{M}$$

is necessary to guarantee the equivalence of P_0 and P_1.

Outline

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Sample Result

- Consider now the general setting where our dictionary Φ is *any* spanning set.
- Set $G := \Phi^\star\Phi$, the *Gram matrix* of Φ.
- Define $M(G) := \max_{j\neq k} |G_{j,k}|$.
- Suppose f has the following representation:

$$f = \Phi\alpha = \sum_{\gamma} \alpha_\gamma \varphi_\gamma$$

Theorem (Donoho and Elad 2003)

If

$$\|\alpha\|_0 < \frac{1}{2}\left(1 + \frac{1}{M(G)}\right),$$

then α is the unique solution to both P_0 and P_1.

Outline

Compressed Sensing

- Recall the D-S uncertainty principle:

$$|\text{supp}(f)| + |\text{supp}(\hat{f})| \geq 2\sqrt{N}$$

- In the context of uncertainty principles, compressed sensing seeks to improve the lower bound of $2\sqrt{N}$ by coming up with an uncertainty relation that holds for *almost every* signal f.
- These results are probabilistic.

Compressed Sensing and the Uncertainty Principle

Theorem (Candes, Romberg, and Tao 2005)

Let $T, \Omega \subseteq \mathbb{Z}/N\mathbb{Z}$ *be selected uniformly at random, and let* $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$ *be a signal such that* $\mathrm{supp}(f) = T$ *and* $\mathrm{supp}(\hat{f}) = \Omega$*. Furthermore, let* ρ *be some pre-determined constant. Then*

$$|\mathrm{supp}(f)| + |\mathrm{supp}(\hat{f})| > \frac{N}{\sqrt{(\rho+1)\log N}}$$

with probability at least $1 - O(N^{-\rho})$*.*

Compressed Sensing and Sparse Representation

- Consider the dictionary $\Phi = \{\text{spikes}\} \cup \{\text{sinusoids}\}$.
- Let $T \subseteq \{(1,\tau)\}$ and $\Omega \subseteq \{(2,\omega)\}$.
- Suppose $f \in \ell^2(\mathbb{Z}/N\mathbb{Z})$ has the following representation:

$$f = \Phi\alpha = \sum_{\gamma\in T} \alpha_\gamma \varphi_\gamma + \sum_{\gamma\in\Omega} \alpha_\gamma \varphi_\gamma.$$

Theorem (Candes and Romberg 2006)

If

$$\|\alpha\|_0 = |T| + |\Omega| \leq \text{Const.} \left(\frac{N}{\log N}\right),$$

then, with high probability, α is the unique solution to both P_0 and P_1.

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