Quasi absolutely minimal Lipschitz extensions

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Overview

1. History of Absolutely Minimal Lipschitz Extensions
   - Extension of Functions
   - Locally Best Extensions
   - Past and Present Results

2. Quasi Absolutely Minimal Lipschitz Extensions
   - Generalized Lipschitz Extensions
   - Existence of Quasi-AMLEs
   - Sketch of Proof
Overview

1. History of Absolutely Minimal Lipschitz Extensions
   - Extension of Functions
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2. Quasi Absolutely Minimal Lipschitz Extensions
The Broad View

- Let $\mathcal{F}(X, Z)$ be some space of functions mapping the set $X$ to the set $Z$ that is endowed with a norm or seminorm $\| \cdot \|$.
- Let $E \subset X$ and $f : E \to Z$.

**Question 1:** Can one extend $f$ to a function $F \in \mathcal{F}(X, Z)$? That is, can one find an $F : X \to Z$ such that
  - $F(x) = f(x)$ for all $x \in E$.
  - $F \in \mathcal{F}(X, Z)$ with $\| F \| < \infty$.

**Question 2:** If we can extend $f$, how small can we make $\| F \|$? Is it possible to find a minimal extension $F \in \mathcal{F}(X, Z)$ such that

$$
\| F \| = \inf \left\{ \| \widetilde{F} \| : \widetilde{F}|_E = f, \widetilde{F} \in \mathcal{F}(X, Z) \right\}.
$$

**Question 3:** If a minimal extension exists, is it unique? If it is not unique, what is the “best” minimal extension?
Lipschitz Functions

Notation throughout the talk:
- \((X, d_X)\) (domain) and \((Z, d_Z)\) (range) are metric spaces.
- \(E \subset X\) is closed.
- \(f : E \to Z\) is a function we wish to extend.
- \(g : D \to Z, D \subset X\), is a generic function.

Lipschitz Constant

Let \(g : X \to Z\). The \textit{Lipschitz constant} of \(g\) over the set \(D \subset X\) is defined as:

\[
\text{Lip}(g; D) \triangleq \sup_{\substack{x, y \in D \cap X \neq y}} \frac{d_Z(g(x), g(y))}{d_X(x, y)}.
\]
**Isometric Lipschitz Extensions**

**Isometric Extension Property**

Two metric spaces \((X, d_X)\) and \((Z, d_Z)\) are said to have the *isometric extension property* if for any function \(f : E \rightarrow Z\) with \(\text{Lip}(f; E) < \infty\), there exists an extension \(F : X \rightarrow Z\) such that

- \(F(x) = f(x)\) for all \(x \in E\).
- \(\text{Lip}(F; X) = \text{Lip}(f; E)\).

There is also the *isomorphic extension property*, in which we require that

\[ \text{Lip}(F; X) \leq C \cdot \text{Lip}(f; E), \]

where \(C\) is a constant depending only on \((X, d_X)\) and \((Z, d_Z)\).
Examples

Recall:
- \((X, d_X)\) - domain.
- \((Z, d_Z)\) - range.

Example (Isometric Extension Property)

- \((X, d_X) = \mathbb{R}^n\) and \((Z, d_Z) = \mathbb{R}\) (McShane, 1934; Whitney, 1934). Can generalize so that \((X, d_X)\) is arbitrary.
- \((X, d_X) = \mathcal{H}_1\) and \((Z, d_Z) = \mathcal{H}_2\), where \(\mathcal{H}_1\) and \(\mathcal{H}_2\) are Hilbert spaces (Kirszbraun, 1934).
- \((X, d_X)\) is arbitrary and \((Z, d_Z)\) is metrically convex and has the binary intersection property, e.g., \((Z, d_Z) = \ell^\infty_n\).
Recall:

- \((X, d_X)\) - domain.
- \((Z, d_Z)\) - range.

### Example (Isomorphic Extension Property)

For \(\infty > p > 2 > q \geq 1\), then \((X, d_X) = L^p\) and \((Z, d_Z) = L^q\) have the isomorphic extension property with \(C \lesssim \sqrt{\frac{p-1}{q-1}}\) (Ball, 1992; Naor, Peres, Schramm, Sheffield, 2006).

### Example (No Extension Property)

For all \(2 < q < \infty\), \((X, d_X) = L^2\) and \((Z, d_Z) = L^q\) do not have any extension property (Naor, 2001).
Non-Uniqueness of the Minimal Extension

- Assume \((X, d_X)\) and \((Z, d_Z)\) have the isometric extension property.
- For an arbitrary function \(f : E \to Z\) with \(\text{Lip}(f; E) < \infty\), the minimal extension \(F : X \to Z\) is in general not unique.

**Example**

- Set \((X, d_X) = \mathbb{R}^n\) and \((Z, d_Z) = \mathbb{R}\).
- Let \(f : E \to \mathbb{R}\) with \(\text{Lip}(f; E) < \infty\).
- Two minimal extensions of \(f\) are given by:

\[
\begin{align*}
    m(f)(x) &\triangleq \sup_{y \in E} (f(y) - \text{Lip}(f; E)\|x - y\|), \quad x \in \mathbb{R}^n \\
    M(f)(x) &\triangleq \inf_{y \in E} (f(y) + \text{Lip}(f; E)\|x - y\|), \quad x \in \mathbb{R}^n
\end{align*}
\]

- In general, \(m(f) \neq M(f)\), and there is a range of minimal extensions \(F : \mathbb{R}^n \to \mathbb{R}\) satisfying \(m(f) \leq F \leq M(f)\).
Non-Uniqueness of the Minimal Extension

Example (continued)

- $n = 1$, so that $(X, d_X) = \mathbb{R}$.
- $E = \{-1, 0, 1\}$.
- $f(-1) = 0, f(0) = 0, f(1) = 1$. 

![Graph showing the function $f$ and its extension $\tilde{f}$]
Absolutely Minimal Lipschitz Extensions

Absolutely Minimal Lipschitz Extension

Let $f : E \to Z$ have minimal Lipschitz extension $F : X \to Z$ such that $\text{Lip}(F; X) = \text{Lip}(f; E)$. The function $F$ is an absolutely minimal Lipschitz extension (AMLE) if for every open subset $V \subset X \setminus E$ and every Lipschitz mapping $\tilde{F} : X \to Z$ that coincides with $F$ on $X \setminus V$,

$$\text{Lip}(F; V) \leq \text{Lip}(\tilde{F}; V).$$

An AMLE is the “locally best” Lipschitz extension.
Absolutely Minimal Lipschitz Extensions

When \((X, d_X)\) is path connected, the following definition of an AMLE is equivalent to the previous one.

**Absolutely Minimal Lipschitz Extension (Aronsson, 1967)**

Let \(f : E \to Z\) have minimal Lipschitz extension \(F : X \to Z\) such that \(\text{Lip}(F; X) = \text{Lip}(f; E)\). The function \(F\) is an *absolutely minimal Lipschitz extension (AMLE)* if for every open subset \(V \subset X \setminus E\),

\[
\text{Lip}(F; V) = \text{Lip}(F; \partial V).
\]
Back to the Example

Example

- \((X, d_X) = (Z, d_Z) = \mathbb{R}\).
- \(E = \{-1, 0, 1\}\).
- \(f(-1) = 0, f(0) = 0, f(1) = 1\).
Existence and Uniqueness

- Let \((X, d_X) = \mathbb{R}^n\).
- Let \((Z, d_Z) = \mathbb{R}\).

Let \(f : E \to \mathbb{R}\) with \(\text{Lip}(f; E) < \infty\). Then,

- **Existence:** An AMLE extending \(f\) exists (Aronsson, 1967).
- **Uniqueness:** The AMLE extending \(f\) is unique (Jensen, 1993).
Relationship to PDEs

The Infinity Laplacian

\[ \Delta_\infty g \triangleq \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}. \]

Equivalence

Given \( f : E \to \mathbb{R} \) with \( \text{Lip}(f; E) < \infty \), let \( F : \mathbb{R}^n \to \mathbb{R} \) be a minimal Lipschitz extension of \( f \).

- If \( F \in C^2 \), then \( F \) is the AMLE for \( f \) \( \iff \Delta_\infty F = 0 \) on \( \mathbb{R}^n \setminus E \) (Aronsson, 1967).
- If \( F \notin C^2 \), then \( F \) is an AMLE for \( f \) \( \iff \Delta_\infty F = 0 \) on \( \mathbb{R}^n \setminus E \), interpreted as a viscosity solution (Jensen, 1993).
Generalizations on the Domain

- Let \((X, d_X)\) be a length space.
- Set \((Z, d_Z) = \mathbb{R}\).

## Existence and Uniqueness

### Existence of an AMLE:
- Mil’man, 1999.
- Juutinen, 2002 \([\langle X, d_X \rangle \text{ is separable}]\).
- Le Gruyer, 2007 \([\langle X, d_X \rangle \text{ is compact}]\).

### Uniqueness of the AMLE:
- Peres, Schramm, Sheffield, an Wilson , 2009 (Tug of War).
Results for Non-Scalar Valued Functions

Naor and Sheffield, 2012

AMLEs exist and are unique when:
- \((X, d_X)\) is a locally compact length space.
- \((Z, d_Z)\) is a metric tree.
- Politics.

Sheffield and Smart, 2012

Tight AMLEs exist and are unique when:
- \((X, d_X)\) is a finite graph.
- \((Z, d_Z) = \mathbb{R}^m\).

Also addresses the case when \((X, d_X) = \mathbb{R}^n\) (but does not solve it).
Overview

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Properties of the Metric Spaces

For the domain \((X, d_X)\), we require:

- Compact.
- Every two points are joined by a unique geodesic of finite length. Note that this implies that for every two points \(x, y \in X\), there exists a unique midpoint \(m(x, y) \in X\) such that

\[
d_X(x, m(x, y)) = d_X(m(x, y), y) = \frac{1}{2} d_X(x, y).
\]

For the range \((Z, d_Z)\):

- Complete.
Set \( \mathcal{F}(X, Z) \triangleq \{ g : D \to Z : D \subset X \} \).

A generalized Lipschitz functional is a functional \( \Phi \) of the form:

\[
\Phi : \mathcal{F}(X, Z) \to \mathcal{F}(X \times X, \mathbb{R}^+ \cup \{\infty\})
\]

\[
g \mapsto \Phi(g; \cdot, \cdot) : \text{dom}(g) \times \text{dom}(g) \to \mathbb{R}^+ \cup \{\infty\}.
\]

We also set, for any \( D \subset \text{dom}(g) \),

\[
\Phi(g; D) \triangleq \sup_{x, y \in D, x \neq y} \Phi(g; x, y).
\]

Define \( \mathcal{F}_{\Phi}(X, Z) = \{ g \in \mathcal{F}(X, Z) : \Phi(g; \text{dom}(g)) < \infty \} \).
Generalized Lipschitz Functionals

Example

\[ \Phi(g; x, y) = \frac{d_Z(g(x), g(y))}{d_X(x, y)}, \]

\[ \Phi(g; D) = \sup_{x, y \in D} \Phi(g; x, y) = \operatorname{Lip}(g; D). \]
Quasi absolutely minimal Lipschitz extensions

Generalized Lipschitz Functionals

Minimal and Absolutely Minimal Lipschitz Extensions

Consider \( f : E \to Z \) such that \( f \in \mathcal{F}_\Phi(X, Z) \). A function \( F : X \to Z \) is a minimal extension of \( f \) if:

- \( F(x) = f(x) \) for all \( x \in E \).
- \( \Phi(F; X) = \Phi(f; E) \).

The function \( F \) is an AMLE for \( f \) if it additionally satisfies

\[
\Phi(F; V) = \Phi(F; \partial V), \quad \text{for all open } V \subset X \setminus E.
\]

Note, this is equivalent to the analogue of our original definition. That is, for every open subset \( V \subset X \setminus E \) and every \( \Phi \)-Lipschitz mapping \( \tilde{F} : X \to Z \) that coincides with \( F \) on \( X \setminus V \),

\[
\Phi(F; V) \leq \Phi(\tilde{F}; V).
\]
Properties of $\Phi$

1. **Symmetry:** $\Phi(g; x, y) = \Phi(g; y, x)$.

2. **Isometric extension property:** For all $g \in \mathcal{F}_\Phi(X, Z)$ with $\text{dom}(g) = D$, there exists an extension $G : X \to Z$ such that $\Phi(G; X) = \Phi(g; D)$.

3. **Continuity of the function:** If $g \in \mathcal{F}_\Phi(X, Z)$, then $g$ is a continuous function.

4. **Continuity of the functional:** Let $g \in \mathcal{F}_\Phi(X, Z)$ with $\text{dom}(g) = D$. For each $x, y \in D$, there exists $\eta > 0$ such that

$$\forall z \in B_\eta(y) \cap D, \quad |\Phi(g; x, y) - \Phi(g; x, z)| < \varepsilon.$$
Properties of $\Phi$

- Set $B_{1/2}(x, y) \triangleq B_r(m(x, y))$, with $r = \frac{1}{2}d_X(x, y)$.
- For each $x, y \in X$, let $\Gamma(x, y)$ denote the set of curves $\gamma : [0, 1] \to B_{1/2}(x, y)$ such that $\gamma(0) = x$, $\gamma(1) = y$, $\gamma$ is continuous, and $\gamma$ is monotone in the following sense:

  $$\text{if } 0 \leq t < s \leq 1, \text{ then } d_X(x, \gamma(t)) < d_X(x, \gamma(s)).$$

Final property of $\Phi$:

5 **Bounding Curve:** For all $g \in \mathcal{F}_\Phi(X, Z)$ with $\text{dom}(g) = D$, and for all $x, y \in E$ with $B_{1/2}(x, y) \subset D$, there exists a curve $\gamma \in \Gamma(x, y)$ such that

$$\Phi(g; x, y) \leq \inf_{t \in [0,1]} \max \{\Phi(g; x, \gamma(t)), \Phi(g; \gamma(t), y)\}.$$
Examples of Admissible Triplets \((X, Z, \Phi)\)

Example

1. \(\Phi(g; D) = \text{Lip}(g; D)\) and any pair \((X, d_X)\) and \((Z, d_Z)\) that have the isometric extension property.

2. Let \(\alpha \in (0, 1]\), and define the Lipschitz-Hölder constant as

\[
\text{Lip}_\alpha(g; D) \triangleq \sup_{x,y \in D, x \neq y} \frac{d_Z(g(x), g(y))}{d_X(x, y)^\alpha}.
\]

Then \(\Phi(g; D) = \text{Lip}_\alpha(g; D)\) and any pair \((X, d_X)\) and \((Z, d_Z)\) that have the isometric extension property for \(\text{Lip}_\alpha\). Some examples of such pairs are:

- \((X, d_X) = (Z, d_Z) = \mathcal{H}\), where \(\mathcal{H}\) is a Hilbert space, and \(0 < \alpha \leq 1\).
- \((X, d_X)\) arbitrary, \((Z, d_Z) = L^p(\mathcal{N}, \nu)\), and:
  - \(1 < p \leq 2\) with \(0 < \alpha \leq \frac{p-1}{p}\).
  - \(2 \leq p < \infty\) with \(0 < \alpha \leq \frac{1}{p}\).
Examples of Admissible Triplets \((X, Z, \Phi)\)

Example (1-Fields, Le Gruyer, 2009)

3 Set \((X, d_X) = \mathbb{R}^n\) and let \((Z, d_Z) = \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})\), the set of 1st order polynomials (affine functions).

- Notation:

\[
T : \mathbb{R}^n \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})
\]
\[
x \mapsto T_x
\]

- Define \(\Phi\) as:

\[
\Phi(T; x, y) = 2 \sup_{a \in \mathbb{R}^n} \frac{|T_x(a) - T_y(a)|}{\|x - a\|^2 + \|y - a\|^2}.
\]

- Meaning of the minimal extension: Let \(T : E \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})\), and let \(U : \mathbb{R}^n \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})\) be a minimal extension of \(T\). Set \(F(x) \triangleq U_x(x)\).

- Extension: \(F\) extends \(T\) so that

\[
J_xF(a) \triangleq F(x) + \nabla F(x) \cdot (a - x) = T_x(a) \text{ for all } x \in E.
\]

- Minimal: \(\text{Lip}(\nabla F)\) is minimal amongst all such extensions.
Let $N_0 \in \mathbb{N}$.

Let $\rho > 0$.

**Set of Open Sets**

$$\mathcal{O}(\rho, N_0) \triangleq \left\{ \Omega = \bigcup_{i=1}^{N} B_{r_i}(x_i) : x_i \in X, \ r_i \geq \rho, \ N \leq N_0 \right\}.$$
2\textsuperscript{nd} Approximation: The Lipschitz Functional

- Let $\alpha \geq 0$.
- Let $g : D \to Z$, $D \subset X$.
- Let $V \subset D$ be open.

**Approximation Functional**

$$
\Psi_\alpha(f; V) \triangleq \sup \left\{ \Phi(f; x, y) : B_{d_X(x,y)}(x) \subset V, \ d_X(x, \partial V) \geq \alpha \right\}.
$$
Main Result

**Theorem (H. and Le Gruyer, 2012)**

Given an admissible triple \((X, Z, \Phi)\), as well as \(f \in \mathcal{F}_\Phi(X, Z)\) with \(\text{dom}(f) = E, \rho > 0, N_0 \in \mathbb{N}, \alpha > 0,\) and \(\sigma_0 > 0\), there exists a *quasi-AMLE* \(F : X \rightarrow Z\) that satisfies:

1. \(F\) is a minimal extension of \(f\), i.e.,
   - \(F(x) = f(x)\) for all \(x \in E\).
   - \(\Phi(F; X) = \Phi(f; E)\).

2. The following quasi-AMLE condition is satisfied:
   
   \[
   \Psi_\alpha(F; \Omega) - \Phi(F; \partial \Omega) < \sigma_0, \quad \forall \Omega \in \mathcal{O}(\rho, N_0), \quad \Omega \subset X \setminus E.
   \]

**Proposition (H. and Le Gruyer, 2012)**

\[
\Phi(f; V) = \max \{\Psi_0(f; V), \Phi(f; \partial V)\}, \quad \forall \text{ open } V \subset X.
\]
Let $g \in \mathcal{F}_\Phi(X, Z)$ with $\text{dom}(g) = D$.

Let $V \subset X$ be open such that $\overline{V} \subset D$.

**Correction Operator**

Use the isometric extension property of $\Phi$ to obtain $G : \overline{V} \to Z$ such that

- $G(x) = g(x)$ for all $x \in \partial V$.
- $\Phi(G; V) = \Phi(g; \partial V)$.

Define the correction operator $H$ as:

$$H(g; V)(x) \triangleq G(x).$$
We construct a sequence of minimal extensions 
\( \{F_i : X \rightarrow Z : i \in \mathbb{N} \} \) for \( f : E \rightarrow Z \).

For each \( i \in \mathbb{N} \), define:

\[
\Delta_i \triangleq \{ \Omega \in \mathcal{O}(\rho, N_0) : \Psi_\alpha(F_i; \Omega) - \Phi(F_i; \partial \Omega) \geq \sigma_0, \ \Omega \subset X \setminus E \}.
\]

If \( \Delta_i = \emptyset \), then \( U_i \) is a quasi-AMLE for \( f \).

If \( \Delta_i \neq \emptyset \), then pick any \( \Omega_{i+1} \in \Delta_i \) and construct \( F_{i+1} \):

\[
F_{i+1}(x) \triangleq \begin{cases} 
H(F_i; \Omega_{i+1})(x), & x \in \Omega_{i+1}, \\
F_i(x), & x \in X \setminus \Omega_{i+1}.
\end{cases}
\]
Main Lemma

Main Lemma (H. and Le Gruyer, 2012)

The following property holds true for all $p \in \mathbb{N}$:

$$\exists M_p \in \mathbb{N} \quad \text{s.t.} \quad \forall i > M_p, \quad \Phi(F_i; \Omega_i) < \Phi(f; E) - p \frac{\sigma_0}{2}. \quad (Q_p)$$

- **Initial Case:** $p = 1$

  $$\Phi(F_i; \Omega_i) = \Phi(H(F_{i-1}; \Omega_i); \Omega_i)$$
  $$= \Phi(F_{i-1}; \partial \Omega_i)$$
  $$\leq \Psi_\alpha(F_{i-1}; \Omega_i) - \sigma_0$$
  $$\leq \Phi(f; E) - \sigma_0.$$  

- **Inductive Step:** Harder...
Questions

- Given a sequence of quasi-AMLEs \( \{ F_{\rho,N_0,\alpha,\sigma_0} \} \), can we take the limit \( F_{\rho,N_0,\alpha,\sigma_0} \rightarrow F_0 \) as \( \rho, \frac{1}{N_0}, \alpha, \sigma_0 \rightarrow 0 \) to obtain an actual AMLE?

- PDE or stochastic game analogues for 1-fields?

- What if we only have the isomorphic extension property?
Thank you

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