Quasi absolutely minimal Lipschitz extensions

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Collaborators

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Overview

Overview

The Broad View

- Let $\mathbb{F}(X, Z)$ be some space of functions mapping the set X to the set Z that is endowed with a norm or seminorm $\|\cdot\|$.
- Let $E \subset X$ and $f : E \to Z$.

- Question 1: Can one extend f to a function $F \in \mathbb{F}(X, Z)$? That is, can one find an $F : X \to Z$ such that
  - $F(x) = f(x)$ for all $x \in E$.
  - $F \in \mathbb{F}(X, Z)$ with $\|F\| < \infty$.
- Question 2: If we can extend f, how small can we make $\|F\|$? Is it possible to find a minimal extension $F \in \mathbb{F}(X, Z)$ such that

$$\|F\| = \inf \left\{ \|\widetilde{F}\| : \widetilde{F}|_E = f, \, \widetilde{F} \in \mathbb{F}(X, Z) \right\}.$$

- Question 3: If a minimal extension exists, is it unique? If it is not unique, what is the "best" minimal extension?

Lipschitz Functions

Notation throughout the talk:

- (X, d_X) (domain) and (Z, d_Z) (range) are metric spaces.
- $E \subset X$ is closed.
- $f : E \to Z$ is a function we wish to extend.
- $g : D \to Z$, $D \subset X$, is a generic function.

Lipschitz Constant

Let $g : X \to Z$. The *Lipschitz constant* of g over the set $D \subset X$ is defined as:

$$\mathrm{Lip}(g; D) \triangleq \sup_{\substack{x,y \in D \\ x \neq y}} \frac{d_Z(g(x), g(y))}{d_X(x, y)}.$$

Isometric Lipschitz Extensions

Isometric Extension Property

Two metric spaces (X, d_X) and (Z, d_Z) are said to have the *isometric extension property* if for any function $f : E \to Z$ with $\mathrm{Lip}(f; E) < \infty$, there exists an extension $F : X \to Z$ such that

- $F(x) = f(x)$ for all $x \in E$.
- $\mathrm{Lip}(F; X) = \mathrm{Lip}(f; E)$.

There is also the *isomorphic extension property*, in which we require that

$$\mathrm{Lip}(F; X) \leq C \cdot \mathrm{Lip}(f; E),$$

where C is a constant depending only on (X, d_X) and (Z, d_Z).

Examples

Recall:

- (X, d_X) - domain.
- (Z, d_Z) - range.

Example (Isometric Extension Property)

- $(X, d_X) = \mathbb{R}^n$ and $(Z, d_Z) = \mathbb{R}$ (McShane, 1934; Whitney, 1934). Can generalize so that (X, d_X) is arbitrary.
- $(X, d_X) = \mathcal{H}_1$ and $(Z, d_Z) = \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces (Kirszbraun, 1934).
- (X, d_X) is arbitrary and (Z, d_Z) is metrically convex and has the binary intersection property, e.g., $(Z, d_Z) = \ell_n^\infty$.

Recall:

- (X, d_X) - domain.
- (Z, d_Z) - range.

Example (Isomorphic Extension Property)

- For $\infty > p > 2 > q \geq 1$, then $(X, d_X) = L^p$ and $(Z, d_Z) = L^q$ have the isomorphic extension property with $C \lesssim \sqrt{\frac{p-1}{q-1}}$ (Ball, 1992; Naor, Peres, Schramm, Sheffield, 2006).

Example (No Extension Property)

- For all $2 < q < \infty$, $(X, d_X) = L^2$ and $(Z, d_Z) = L^q$ do not have any extension property (Naor, 2001).

Non-Uniqueness of the Minimal Extension

- Assume (X, d_X) and (Z, d_Z) have the isometric extension property.
- For an arbitrary function $f : E \to Z$ with $\mathrm{Lip}(f; E) < \infty$, the minimal extension $F : X \to Z$ is in general *not* unique.

Example

- Set $(X, d_X) = \mathbb{R}^n$ and $(Z, d_Z) = \mathbb{R}$.
- Let $f : E \to \mathbb{R}$ with $\mathrm{Lip}(f; E) < \infty$.
- Two minimal extensions of f are given by:

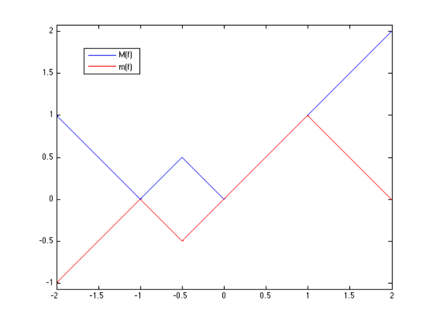
$$m(f)(x) \triangleq \sup_{y \in E} \left(f(y) - \mathrm{Lip}(f; E)\|x - y\|\right), \quad x \in \mathbb{R}^n$$
$$M(f)(x) \triangleq \inf_{y \in E} \left(f(y) + \mathrm{Lip}(f; E)\|x - y\|\right), \quad x \in \mathbb{R}^n$$

- In general, $m(f) \neq M(f)$, and there is a range of minimal extensions $F : \mathbb{R}^n \to \mathbb{R}$ satisfying $m(f) \leq F \leq M(f)$.

Non-Uniqueness of the Minimal Extension

Example (continued)

- $n = 1$, so that $(X, d_X) = \mathbb{R}$.
- $E = \{-1, 0, 1\}$.
- $f(-1) = 0, f(0) = 0, f(1) = 1$.



Absolutely Minimal Lipschitz Extensions

Absolutely Minimal Lipschitz Extension

Let $f : E \to Z$ have minimal Lipschitz extension $F : X \to Z$ such that $\mathrm{Lip}(F; X) = \mathrm{Lip}(f; E)$. The function F is an *absolutely minimal Lipschitz extension (AMLE)* if for every open subset $V \subset X \setminus E$ and every Lipschitz mapping $\widetilde{F} : X \to Z$ that coincides with F on $X \setminus V$,

$$\mathrm{Lip}(F; V) \leq \mathrm{Lip}(\widetilde{F}; V).$$

- An AMLE is the “locally best” Lipschitz extension.

Absolutely Minimal Lipschitz Extensions

- When (X, d_X) is path connected, the following definition of an AMLE is equivalent to the previous one.

Absolutely Minimal Lipschitz Extension (Aronsson, 1967)

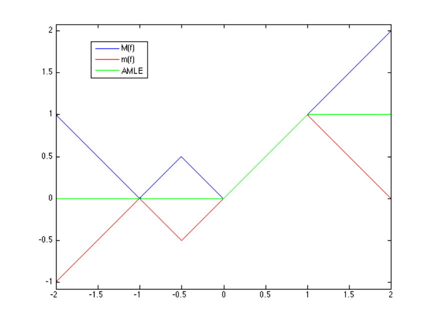
Let $f : E \to Z$ have minimal Lipschitz extension $F : X \to Z$ such that $\mathrm{Lip}(F; X) = \mathrm{Lip}(f; E)$. The function F is an *absolutely minimal Lipschitz extension (AMLE)* if for every open subset $V \subset X \setminus E$,

$$\mathrm{Lip}(F; V) = \mathrm{Lip}(F; \partial V).$$

Back to the Example

Example

- $(X, d_X) = (Z, d_Z) = \mathbb{R}$.
- $E = \{-1, 0, 1\}$.
- $f(-1) = 0, f(0) = 0, f(1) = 1$.



Existence and Uniqueness

- Let $(X, d_X) = \mathbb{R}^n$.
- Let $(Z, d_Z) = \mathbb{R}$.

Existence and Uniqueness

Let $f : E \to \mathbb{R}$ with $\mathrm{Lip}(f; E) < \infty$. Then,

- Existence: An AMLE extending f exists (Aronsson, 1967).
- Uniqueness: The AMLE extending f is unique (Jensen, 1993).

Relationship to PDEs

The Infinity Laplacian

$$\Delta_\infty g \triangleq \sum_{i,j=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j} \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Equivalence

Given $f : E \to \mathbb{R}$ with $\mathrm{Lip}(f;E) < \infty$, let $F : \mathbb{R}^n \to \mathbb{R}$ be a minimal Lipschitz extension of f.

- If $F \in C^2$, then F is the AMLE for $f \iff \Delta_\infty F = 0$ on $\mathbb{R}^n \setminus E$ (Aronsson, 1967).
- If $F \notin C^2$, then F is an AMLE for $f \iff \Delta_\infty F = 0$ on $\mathbb{R}^n \setminus E$, interpreted as a viscosity solution (Jensen, 1993).

Generalizations on the Domain

- Let (X, d_X) be a length space.
- Set $(Z, d_Z) = \mathbb{R}$.

Existence and Uniqueness

Existence of an AMLE:

- Mil'man, 1999.
- Juutinen, 2002 [(X, d_X) is separable].
- Le Gruyer, 2007 [(X, d_X) is compact].

Uniqueness of the AMLE:

- Peres, Schramm, Sheffield, an Wilson , 2009 (Tug of War).

Results for Non-Scalar Valued Functions

Naor and Sheffield, 2012

AMLEs exist and are unique when:

- (X, d_X) is a locally compact length space.
- (Z, d_Z) is a metric tree.
- Politics.

Sheffield and Smart, 2012

Tight AMLEs exist and are unique when:

- (X, d_X) is a finite graph.
- $(Z, d_Z) = \mathbb{R}^m$.

Also addresses the case when $(X, d_X) = \mathbb{R}^n$ (but does not solve it).

Overview

Properties of the Metric Spaces

For the domain (X, d_X), we require:

- Compact.
- Every two points are joined by a unique geodesic of finite length. Note that this implies that for every two points $x, y \in X$, there exists a unique midpoint $m(x, y) \in X$ such that

$$d_X(x, m(x, y)) = d_X(m(x, y), y) = \frac{1}{2} d_X(x, y).$$

For the range (Z, d_Z):

- Complete.

Generalized Lipschitz Functionals

- Set $\mathcal{F}(X,Z) \triangleq \{\, g : D \to Z : D \subset X \}$.

Generalized Lipschitz Functionals

A *generalized Lipschitz functional* is a functional Φ of the form:

$$\begin{aligned} \Phi : \mathcal{F}(X,Z) &\to \mathcal{F}(X \times X, \mathbb{R}^+ \cup \{\infty\}) \\ g &\mapsto \Phi(g;\cdot,\cdot) : \mathrm{dom}(g) \times \mathrm{dom}(g) \to \mathbb{R}^+ \cup \{\infty\}. \end{aligned}$$

We also set, for any $D \subset \mathrm{dom}(g)$,

$$\Phi(g;D) \triangleq \sup_{\substack{x,y \in D \\ x \neq y}} \Phi(g;x,y).$$

- Define $\mathcal{F}_\Phi(X,Z) = \{\, g \in \mathcal{F}(X,Z) : \Phi(g;\mathrm{dom}(g)) < \infty \}$.

Generalized Lipschitz Functionals

Example

$$\Phi(g;x,y) = \frac{d_Z(g(x),g(y))}{d_X(x,y)},$$

$$\Phi(g;D) = \sup_{\substack{x,y \in D \\ x \neq y}} \Phi(g;x,y) = \mathrm{Lip}(g;D).$$

Generalized Lipschitz Functionals

Minimal and Absolutely Minimal Lipschitz Extensions

Consider $f : E \to Z$ such that $f \in \mathcal{F}_\Phi(X, Z)$. A function $F : X \to Z$ is a *minimal extension* of f if:

- $F(x) = f(x)$ for all $x \in E$.
- $\Phi(F; X) = \Phi(f; E)$.

The function F is an *AMLE* for f if it additionally satisfies

$$\Phi(F; V) = \Phi(F; \partial V), \quad \text{for all open } V \subset X \setminus E.$$

Note, this is equivalent to the analogue of our original definition. That is, for every open subset $V \subset X \setminus E$ and every Φ-Lipschitz mapping $\widetilde{F} : X \to Z$ that coincides with F on $X \setminus V$,

$$\Phi(F; V) \leq \Phi(\widetilde{F}; V).$$

Properties of Φ

1. Symmetry: $\Phi(g; x, y) = \Phi(g; y, x)$.
2. Isometric extension property: For all $g \in \mathcal{F}_\Phi(X, Z)$ with $\mathrm{dom}(g) = D$, there exists an extension $G : X \to Z$ such that $\Phi(G; X) = \Phi(g; D)$.
3. Continuity of the function: If $g \in \mathcal{F}_\Phi(X, Z)$, then g is a continuous function.
4. Continuity of the functional: Let $g \in \mathcal{F}_\Phi(X, Z)$ with $\mathrm{dom}(g) = D$. For each $x, y \in D$, there exists $\eta > 0$ such that

$$\forall z \in B_\eta(y) \cap D, \quad |\Phi(g; x, y) - \Phi(g; x, z)| < \varepsilon.$$

Properties of Φ

- Set $B_{1/2}(x,y) \triangleq B_r(m(x,y))$, with $r = \frac{1}{2}d_X(x,y)$.
- For each $x, y \in X$, let $\Gamma(x,y)$ denote the set of curves $\gamma : [0,1] \to B_{1/2}(x,y)$ such that $\gamma(0) = x$, $\gamma(1) = y$, γ is continuous, and γ is monotone in the following sense:

$$\text{if } 0 \leq t < s \leq 1, \text{ then } d_X(x,\gamma(t)) < d_X(x,\gamma(s)).$$

Final property of Φ:

5. <u>Bounding Curve:</u> For all $g \in \mathcal{F}_\Phi(X,Z)$ with $\mathrm{dom}(g) = D$, and for all $x, y \in E$ with $B_{1/2}(x,y) \subset D$, there exists a curve $\gamma \in \Gamma(x,y)$ such that

$$\Phi(g;x,y) \leq \inf_{t\in[0,1]} \max\{\Phi(g;x,\gamma(t)), \Phi(g;\gamma(t),y)\}.$$

Examples of Admissible Triplets (X, Z, Φ)

Example

1. $\Phi(g;D) = \mathrm{Lip}(g;D)$ and any pair (X, d_X) and (Z, d_Z) that have the isometric extension property.
2. Let $\alpha \in (0,1]$, and define the Lipschitz-Hölder constant as

$$\mathrm{Lip}_\alpha(g;D) \triangleq \sup_{\substack{x,y \in D \\ x \neq y}} \frac{d_Z(g(x), g(y))}{d_X(x,y)^\alpha}.$$

Then $\Phi(g;D) = \mathrm{Lip}_\alpha(g;D)$ and any pair (X, d_X) and (Z, d_Z) that have the isometric extension property for Lip_α. Some examples of such pairs are:

- $(X, d_X) = (Z, d_Z) = \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and $0 < \alpha \leq 1$.
- (X, d_X) arbitrary, $(Z, d_Z) = L^p(\mathcal{N}, \nu)$, and:
  - $1 < p \leq 2$ with $0 < \alpha \leq \frac{p-1}{p}$.
  - $2 \leq p < \infty$ with $0 < \alpha \leq \frac{1}{p}$.

Examples of Admissible Triplets (X, Z, Φ)

Example (1-Fields, Le Gruyer, 2009)

3. Set $(X, d_X) = \mathbb{R}^n$ and let $(Z, d_Z) = \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})$, the set of 1^{st} order polynomials (affine functions).
   - Notation:

$$T : \mathbb{R}^n \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})$$
$$x \mapsto T_x$$

   - Define Φ as:

$$\Phi(T; x, y) = 2 \sup_{a \in \mathbb{R}^n} \frac{|T_x(a) - T_y(a)|}{\|x - a\|^2 + \|y - a\|^2}.$$

   - Meaning of the minimal extension: Let $T : E \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})$, and let $U : \mathbb{R}^n \to \mathcal{P}^1(\mathbb{R}^n, \mathbb{R})$ be a minimal extension of T. Set $F(x) \triangleq U_x(x)$.
     - Extension: F extends T so that $J_x F(a) \triangleq F(x) + \nabla F(x) \cdot (a - x) = T_x(a)$ for all $x \in E$.
     - Minimal: $\mathrm{Lip}(\nabla F)$ is minimal amongst all such extensions.

1^{st} Approximation: Open Sets

- Let $N_0 \in \mathbb{N}$.
- Let $\rho > 0$.

Set of Open Sets

$$\mathcal{O}(\rho, N_0) \triangleq \left\{ \Omega = \bigcup_{i=1}^{N} B_{r_i}(x_i) : x_i \in X, \quad r_i \geq \rho, \quad N \leq N_0 \right\}.$$

2^{nd} Approximation: The Lipschitz Functional

- Let $\alpha \geq 0$.
- Let $g : D \to Z$, $D \subset X$.
- Let $V \subset D$ be open.

Approximation Functional

$$\Psi_\alpha(f; V) \triangleq \sup \left\{ \Phi(f; x, y) : B_{d_X(x,y)}(x) \subset V, \;\; d_X(x, \partial V) \geq \alpha \right\}.$$

Main Result

Theorem (H. and Le Gruyer, 2012)

Given an admissible triple (X, Z, Φ), as well as $f \in \mathcal{F}_{\Phi}(X, Z)$ with $\mathrm{dom}(f) = E$, $\rho > 0$, $N_0 \in \mathbb{N}$, $\alpha > 0$, and $\sigma_0 > 0$, there exists a *quasi-AMLE* $F : X \to Z$ that satisfies:

1. F is a minimal extension of f, i.e.,
   - $F(x) = f(x)$ for all $x \in E$.
   - $\Phi(F; X) = \Phi(f; E)$.
2. The following quasi-AMLE condition is satisfied:

$$\Psi_{\alpha}(F; \Omega) - \Phi(F; \partial\Omega) < \sigma_0, \quad \forall\, \Omega \in \mathcal{O}(\rho, N_0), \;\; \Omega \subset X \setminus E.$$

Proposition (H. and Le Gruyer, 2012)

$$\Phi(f; V) = \max\left\{\Psi_0(f; V), \Phi(f; \partial V)\right\}, \quad \forall \text{ open } V \subset X.$$

Correction Operator

- Let $g \in \mathcal{F}_{\Phi}(X, Z)$ with $\mathrm{dom}(g) = D$.
- Let $V \subset X$ be open such that $\overline{V} \subset D$.

Correction Operator

Use the isometric extension property of Φ to obtain $G : \overline{V} \to Z$ such that

- $G(x) = g(x)$ for all $x \in \partial V$.
- $\Phi(G; V) = \Phi(g; \partial V)$.

Define the correction operator H as:

$$H(g; V)(x) \triangleq G(x).$$

Sequence of Minimal Extensions

- We construct a sequence of minimal extensions $\{F_i : X \to Z : i \in \mathbb{N}\}$ for $f : E \to Z$.
- For each $i \in \mathbb{N}$, define:

$$\Delta_i \triangleq \{\Omega \in \mathcal{O}(\rho, N_0) : \Psi_\alpha(F_i;\Omega) - \Phi(F_i;\partial\Omega) \geq \sigma_0, \;\; \Omega \subset X \setminus E\}\,.$$

- If $\Delta_i = \emptyset$, then U_i is a quasi-AMLE for f.
- If $\Delta_i \neq \emptyset$, then pick any $\Omega_{i+1} \in \Delta_i$ and construct F_{i+1}:

$$F_{i+1}(x) \triangleq \begin{cases} H(F_i;\Omega_{i+1})(x), & x \in \Omega_{i+1}, \\ F_i(x), & x \in X \setminus \Omega_{i+1}. \end{cases}$$

Main Lemma

Main Lemma (H. and Le Gruyer, 2012)

The following property holds true for all $p \in \mathbb{N}$:

$$\exists M_p \in \mathbb{N} \quad \text{s.t.} \quad \forall i > M_p, \;\; \Phi(F_i;\Omega_i) < \Phi(f;E) - p\frac{\sigma_0}{2}. \qquad (\mathbf{Q}_p)$$

- Initial Case: $p = 1$

$$\begin{aligned}\Phi(F_i;\Omega_i) &= \Phi(H(F_{i-1};\Omega_i);\Omega_i)\\ &= \Phi(F_{i-1};\partial\Omega_i)\\ &\le \Psi_\alpha(F_{i-1};\Omega_i) - \sigma_0\\ &\le \Phi(f;E) - \sigma_0.\end{aligned}$$

- Inductive Step: Harder...

Questions

- Given a sequence of quasi-AMLEs $\{F_{\rho,N_0,\alpha,\sigma_0}\}$, can we take the limit $F_{\rho,N_0,\alpha,\sigma_0} \to F_0$ as $\rho, \frac{1}{N_0}, \alpha, \sigma_0 \to 0$ to obtain an actual AMLE?
- PDE or stochastic game analogues for 1-fields?
- What if we only have the isomorphic extension property?

Thank you

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