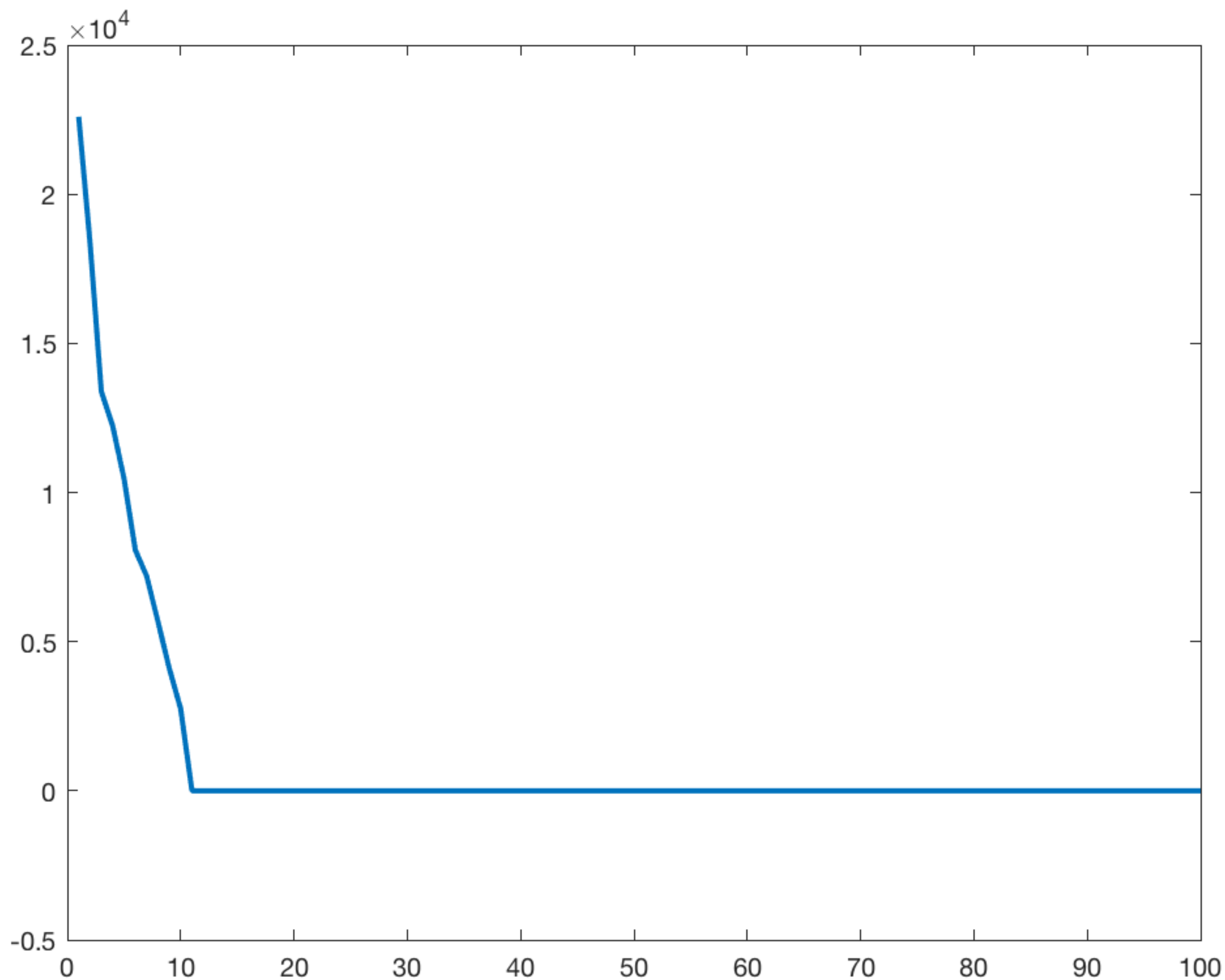
CMSE 820: Mathematical Foundations of Data Science

Lecture 04

PCA denoising

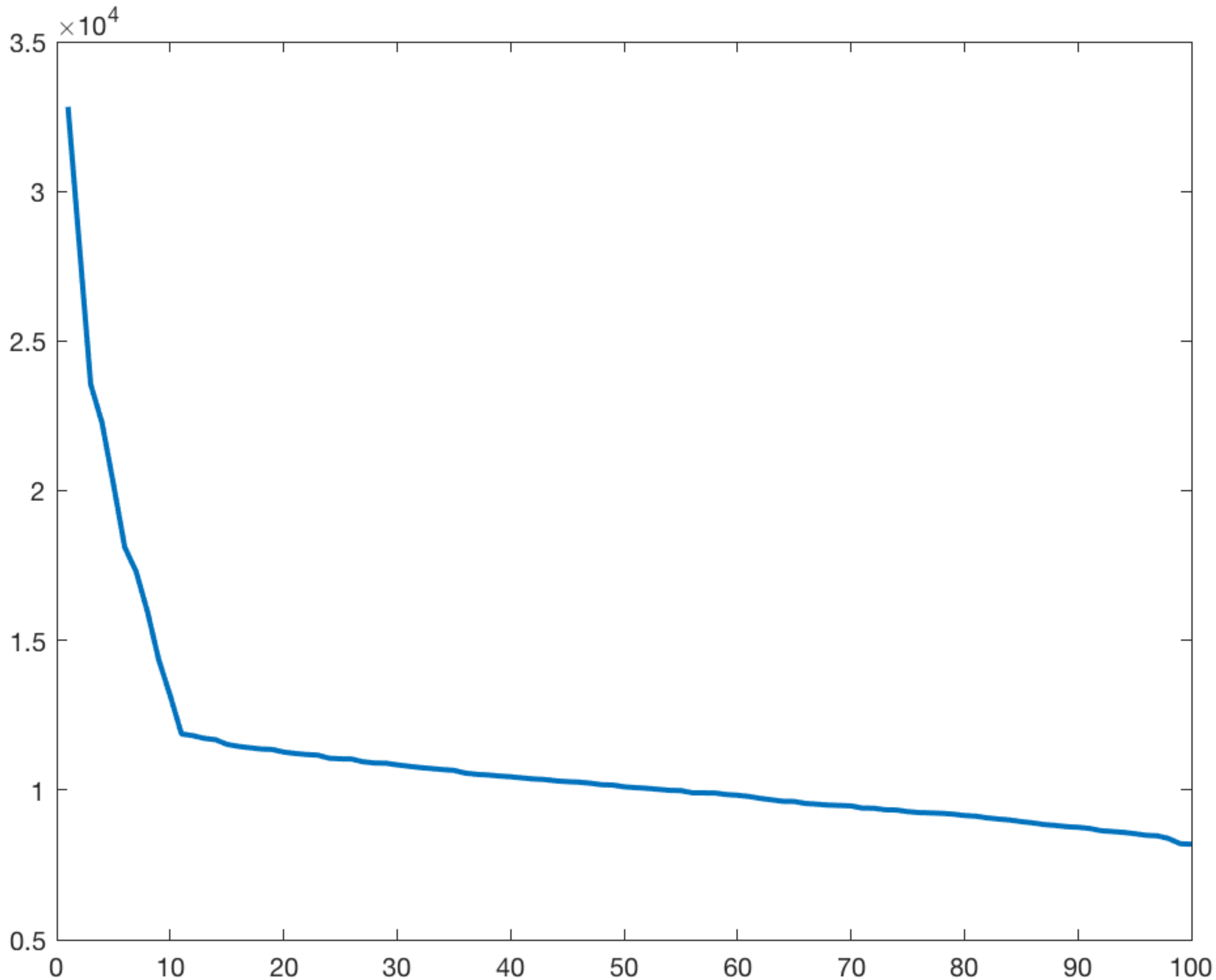
- Data drawn from normal distribution $\mathcal{N}(\Sigma + \epsilon, \mu)$
- $\Sigma \in \mathbb{R}^{p \times p}$ with rank$(\Sigma) = 10$
- $\epsilon \in \mathbb{R}^{p \times p}$ with rank$(\epsilon) = p$, but $\|\epsilon\| \ll \|\Sigma\|$
- Example with $n = 10000$ and $p = 100$

PCA denoising



Without noise

PCA denoising

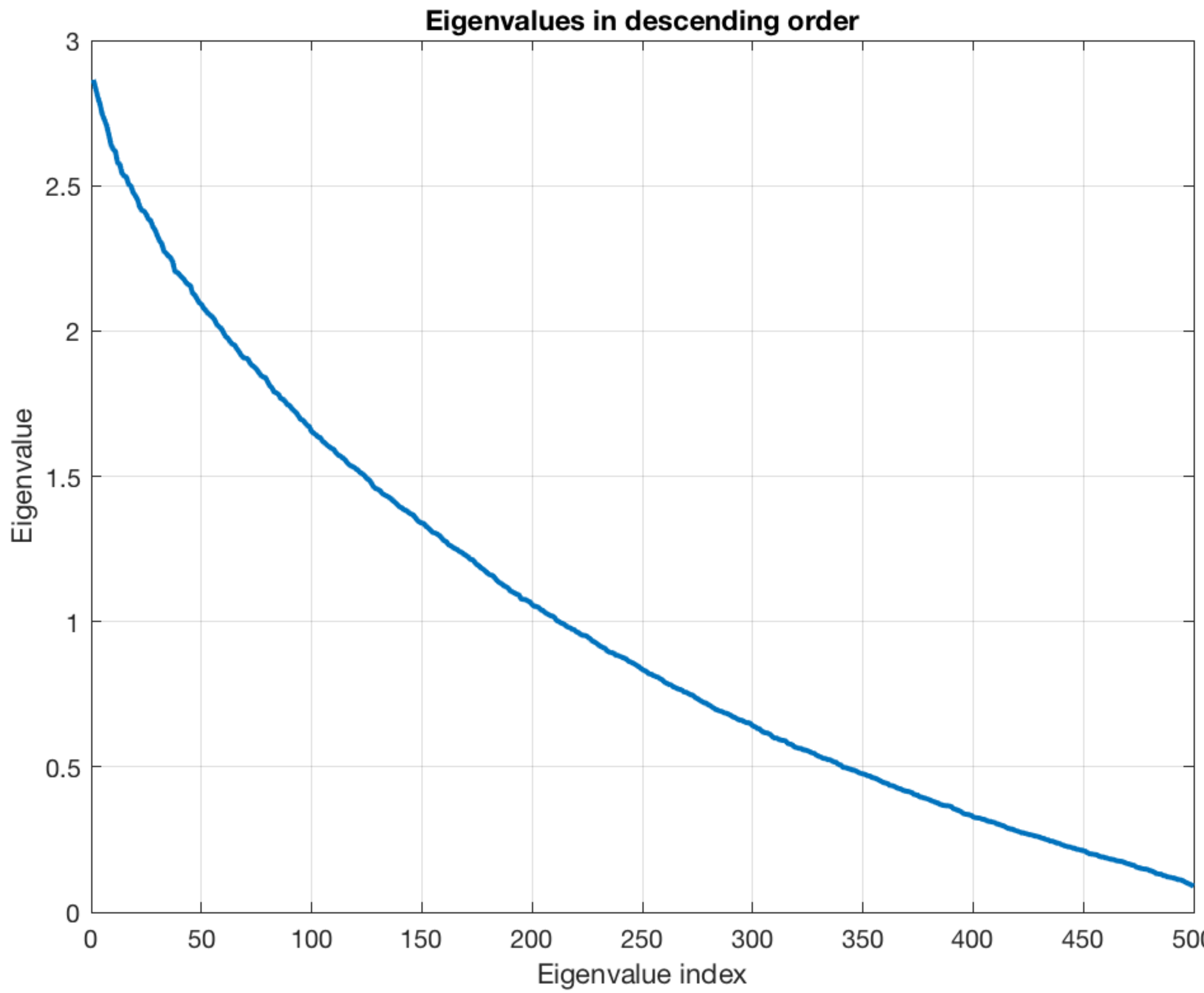


With noise

PCA in high dimensions

- $\mathcal{X}_n = \{x_i\}_{i \leq n} \subset \mathbb{R}^p$
- $X_n = [x_1 \cdots x_n] \in \mathbb{R}^{p \times n}$
- $x_i \sim \mathcal{N}(0, I)$
- $S_n = (1/n) X_n X_n^T$
- Examine spectral properties of S_n when $p, n \to \infty$ and $p/n = \gamma \leq 1$
- In other words, how does PCA perform in high dimensions?

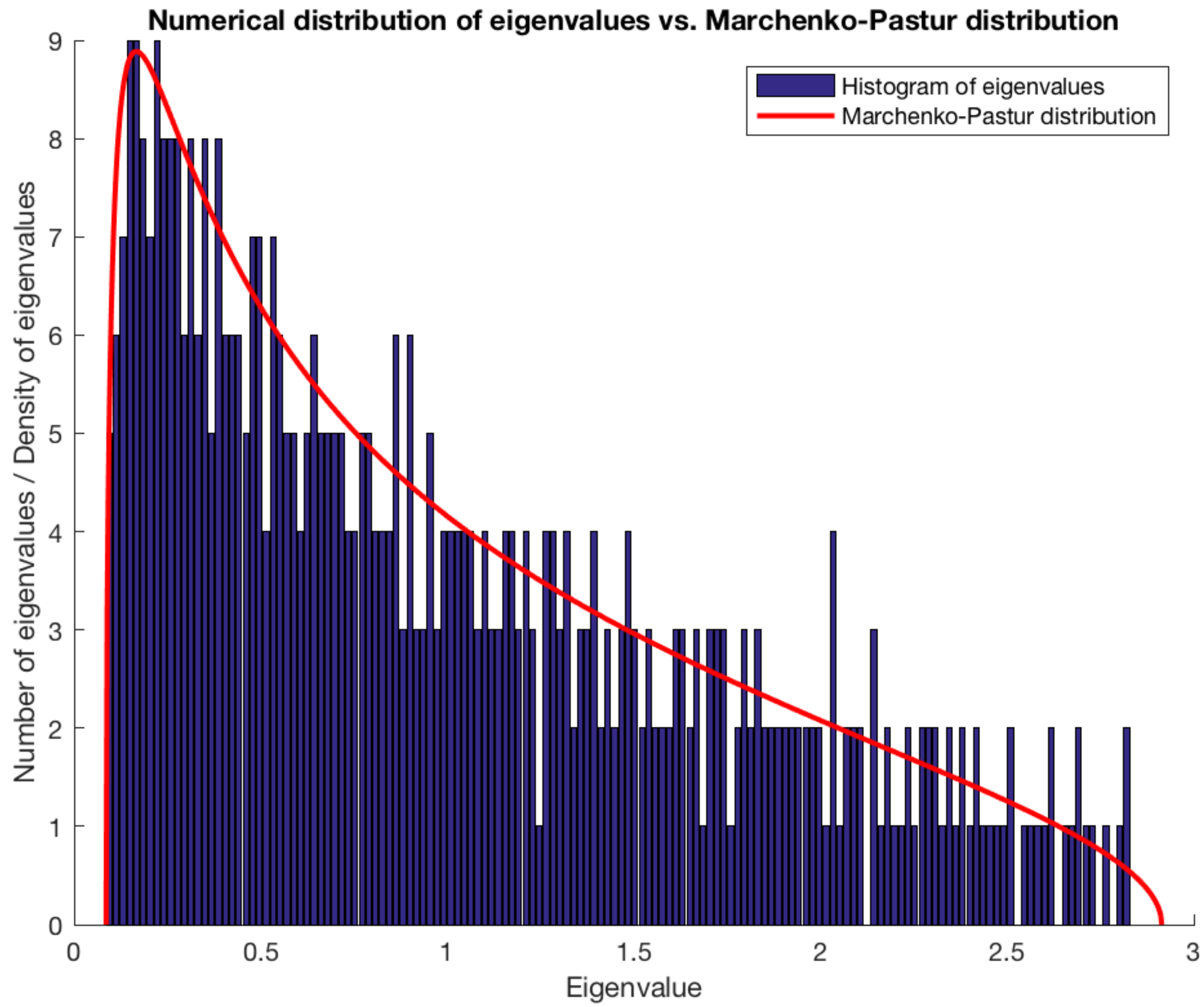
PCA in high dimensions



$p = 500,\ n = 1000$

Eigenvalues of $S_n = (1/n)X_n X_n^T$

PCA in high dimensions



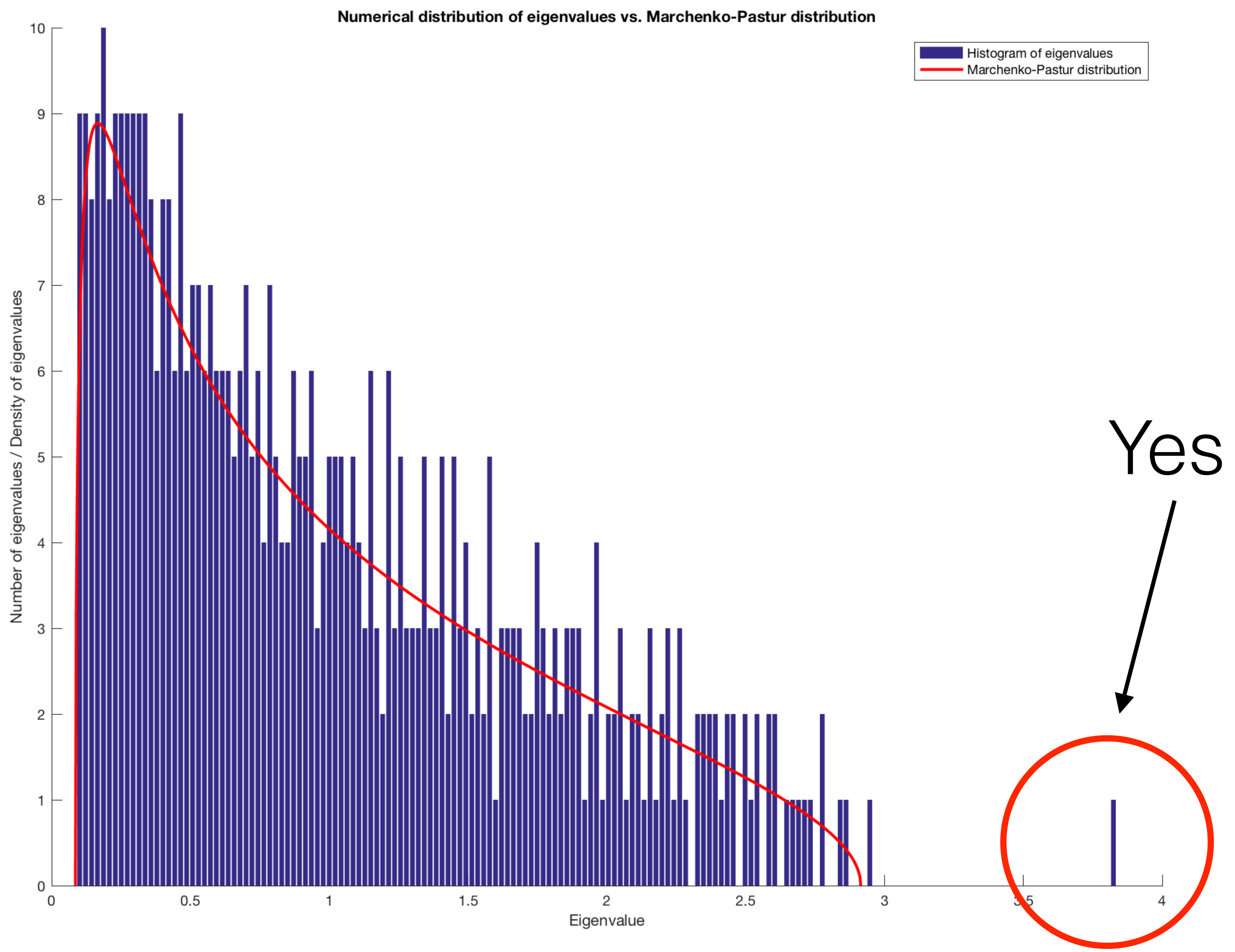
$p = 500,\ n = 1000$

Histogram of eigenvalues of $S_n = (1/n) X_n X_n^T$

Spike model

- $\mathcal{X}_n = \{x_i\}_{i \leq n} \subset \mathbb{R}^p$
- $X_n = [x_1 \cdots x_n] \in \mathbb{R}^{p \times n}$
- $x_i \sim \mathcal{N}(0, I + \beta v v^T)$, where $\beta \geq 0$ and $v \in \mathbb{R}^p$
- $S_n = (1/n) X_n X_n^T$
- Examine spectral properties of S_n when $p, n \to \infty$ and $p/n = \gamma \leq 1$
- Can PCA find the 1-dimensional structure of the data with variance $1 + \beta$ and along the direction v?

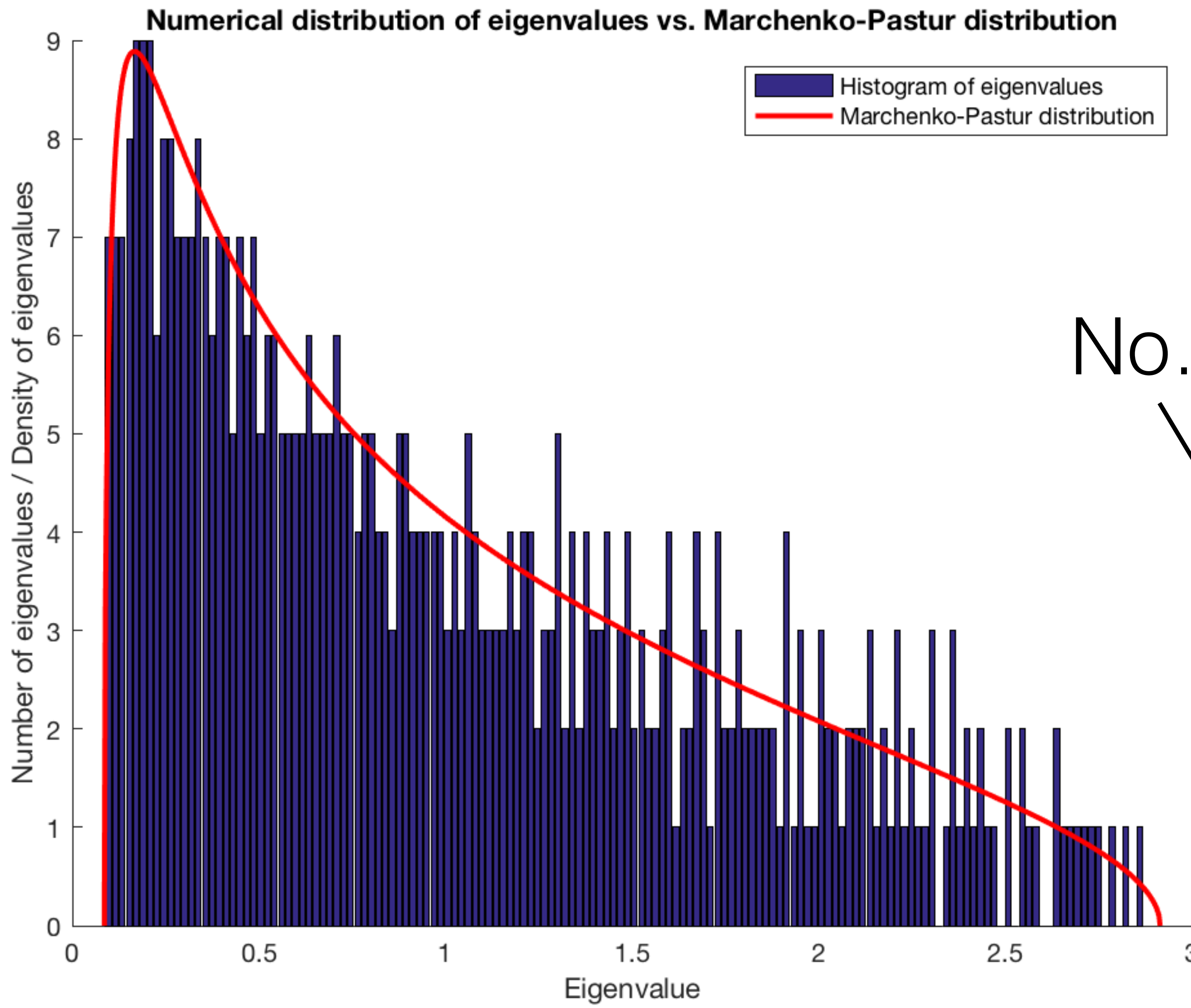
Spike model



$p = 500,\ n = 1000,\ \beta = 2$

Histogram of eigenvalues of $S_n = (1/n)X_n X_n^T$

Spike model



No….

?

$p = 500,\ n = 1000,\ \beta = 1/2$

Histogram of eigenvalues of $S_n = (1/n)X_nX_n^T$