Lecture 16

This is essentially [10, Lecture 2] with small modifications.

8.3.4 The graph Laplacian of some fundamental graphs

We now examine the eigenvalues and eigenvectors of the graph Laplacians of some fundamental graphs on n vertices $V = \{1, \ldots, n\}$:

1. The complete graph K_n, for which $E = \{(i, j) : i \neq j\}$.
2. The star graph S_n, for which $E = \{(1, i) : 2 \leq i \leq n\}$.
3. The ring graph R_n, for which $E = \{(i, i+1) : 1 \leq i < n\} \cup \{(1, n)\}$.
4. The path graph P_n, for which $E = \{(i, i+1) : 1 \leq i < n\}$.

Proposition 3. *The graph Laplacian of K_n has eigenvalue 0 with multiplicity 1 and eigenvalue n with multiplicity $n-1$.*

Proof. Since K_n is connected, the multiplicity of the 0 eigenvalue follows from Proposition 2.

To compute the remaining nonzero eigenvalues, let φ be any non-zero vector orthogonal to $\mathbf{1}$, which implies:

$$0 = \langle \varphi, \mathbf{1} \rangle = \sum_{i=1}^{n} \varphi[i]. \tag{49}$$

Without loss of generality we can take $\varphi[1] \neq 0$. Using (49), we then have:

$$L_{K_n}\varphi[1] = (n-1)\varphi[1] - \sum_{i=2}^{n} \varphi[i] = (n-1)\varphi[1] + \varphi[1] = n\varphi[1].$$

Thus any other eigenvector of L_{K_n} must have eigenvalue n. □

To analyze S_n, we first need the following lemma.

Lemma 1. *Let $G = (V, E)$ be a graph, and let i and j be vertices of degree 1 that are both connected to another vertex k. Then the vector*

$$\varphi[\ell] = \begin{cases} 1 & \ell = i, \\ -1 & \ell = j, \\ 0 & \text{otherwise}, \end{cases} \tag{50}$$

is an eigenvector of L_G with eigenvalue 1.

Proof. Check on your own! □

Note that the existence of the eigenvector φ defined in (50) implies that $\varphi'[i] = \varphi'[j]$ for every other eigenvector φ' with eigenvalue other than 1.

Proposition 4. *The graph Laplacian of S_n has eigenvalue 0 with multiplicity 1, eigenvalue 1 with multiplicity $n-2$, and eigenvalue n with multiplicity 1.*

Proof. Check on your own! Use Proposition 2, Lemma 1, and the fact that

$$\mathrm{Tr}(L_G) = \sum_{i=1}^{n} \lambda_i,$$

for any graph G (and in fact, any $n \times n$ matrix with n eigenvalues). □

Proposition 5. *The graph Laplacian of R_n has eigenvectors*

$$\varphi_k[i] = \sin(2\pi ki/n), \tag{51}$$

$$\psi_k[i] = \cos(2\pi ki/n), \tag{52}$$

for $1 \leq k \leq n/2$. When n is even, $\varphi_{n/2}$ is the all zero vector, so we only have $\psi_{n/2}$. Eigenvectors φ_k and ψ_k have eigenvalue $2 - 2\cos(2\pi k/n)$.

Proof. We are going to use the following trigonometric identity:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

which implies:

$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta.$$

Now observe that:

$$\begin{aligned}
L_{R_n}\varphi_k[i] &= 2\varphi_k[i] - \varphi_k[i-1] - \varphi_k[i+1], \\
&= 2\sin\left(\frac{2\pi ki}{n}\right) - \left[\sin\left(\frac{2\pi ki}{n} - \frac{2\pi k}{n}\right) + \sin\left(\frac{2\pi ki}{n} + \frac{2\pi k}{n}\right)\right], \\
&= 2\sin\left(\frac{2\pi ki}{n}\right) - 2\cos\left(\frac{2\pi k}{n}\right)\sin\left(\frac{2\pi ki}{n}\right), \\
&= \left[2 - 2\cos\left(\frac{2\pi k}{n}\right)\right]\sin\left(\frac{2\pi ik}{n}\right).
\end{aligned}$$

The calculation for ψ_k is similar. □

Proposition 6. *The graph Laplacian of P_n has eigenvectors*

$$\phi_k[i] = \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right),$$

with corresponding eigenvalues:

$$\lambda_k = 2 - 2\cos(\pi k/n).$$

Proof. This proof is more involved than the previous ones. We will derive the eigenvectors and eigenvalues of L_{P_n} by treating P_n as a quotient of R_{2n}. To do so, let i be an arbitrary vertex of P_n; identify it with the vertices i and $2n+1-i$ of R_{2n}.

Now consider the vector:

$$\phi_k[i] = \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right).$$

Notice that:

$$\phi_k[i] = \phi_k[2n+1-i],$$

indeed:

$$\begin{aligned}
\phi_k[2n+i-i] &= \cos\left(\frac{\pi k(2n+1-i)}{n} - \frac{\pi k}{2n}\right),\\
&= \cos\left(2\pi k + \frac{\pi k}{n} - \frac{\pi k i}{n} - \frac{\pi k}{2n}\right),\\
&= \cos\left(-\frac{\pi k i}{n} + \frac{\pi k}{2n}\right),\\
&= \cos\left(\frac{\pi k i}{n} - \frac{\pi k}{2n}\right),\\
&= \phi_k[i].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\phi_k[i] &= \cos\left(\frac{2\pi k i}{2n} - \frac{\pi k}{2n}\right),\\
&= \cos\left(\frac{2\pi k i}{2n}\right)\cos\left(\frac{\pi k}{2n}\right) + \sin\left(\frac{2\pi k i}{2n}\right)\sin\left(\frac{\pi k}{2n}\right),\\
&= \varphi_k[i]\cos\left(\frac{\pi k}{2n}\right) + \psi_k[i]\sin\left(\frac{\pi k}{2n}\right),
\end{aligned}$$

where φ_k and ψ_k are eigenvectors of R_{2n}, defined by (51) and (52), respectively. It follows that ϕ_k is an eigenvector of $L_{R_{2n}}$ with eigenvalue $\lambda_k = 2 - 2\cos(2\pi k/(2n)) = 2 - 2\cos(\pi k/n)$.

Now set:

$$\forall\, 1 \leq i \leq n, \quad v_k[i] = \phi_k[i].$$

We claim that v_k is eigenvector of L_{P_n} with eigenvalue λ_k. To see this, let $1 < i < n$ and compute:

$$\begin{aligned}
L_{P_n} v_k[i] &= 2v_k[i] - v_k[i-1] - v_k[i+1], \\
&= \frac{1}{2}\Big(2\phi_k[i] - \phi_k[i-1] - \phi_k[i+1] + \ldots \\
&\ldots + 2\phi_k[2n+1-i] - \phi_k[2n+1-(i-1)] - \phi_k[2n+1-(i+1)]\Big), \\
&= \frac{1}{2}\Big(L_{R_{2n}}\phi_k[i] + L_{R_{2n}}\phi_k[2n+1-i]\Big), \\
&= \frac{1}{2}\Big(\lambda_k\phi_k[i] + \lambda_k\phi_k[2n+1-i]\Big), \\
&= \lambda_k v_k[i].
\end{aligned}$$

For $i = 1$ we have:

$$\begin{aligned}
L_{P_n} v_k[1] &= v_k[1] - v_k[2], \\
&= 2v_k[1] - v_k[2] - v_k[1], \\
&= 2\phi_k[1] - \phi_k[2] - \phi_k[2n], \\
&= L_{R_{2n}}\phi_k[1], \\
&= \lambda_k v_k[1].
\end{aligned}$$

The calculation for $i = n$ is similar. □

We have now seen that the kth eigenvector of the path graph alternates in sign $k-1$ times. This is consistent with our intuition that the Laplacian of the path graph is a discretization of a continuous string, and that its eigenvectors are approxmations of its fundamental modes of vibration when its ends are free.

If this intuition is correct, then it should continue to be true even if we discretize a string whose material changes along its length. This corresponds to a weighted path graph.

Exercises

Exercise 24. Prove Lemma 1.

Exercise 25. Prove Proposition 4.

Exercise 26. Write a function that takes as inputs the vertices V and edges E of a graph G, and outputs the graph Laplacian L_G and its eigenvectors and eigenvalues. Use this code to verify the results of this lecture on the graphs K_n, S_n, R_n, and P_n (test several different values of n for each graph).

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