4.2.1 Parseval and Inversion for Windowed Fourier

The approach in this section follows the treatment in [7, Chapter 3]. For a more in depth treatment of the windowed Fourier transform and time frequency analysis, [7] is an excellent resource.

Recall that for a window $g \in \mathbf{L}^2(\mathbb{R})$ the windowed Fourier transform of $f \in \mathbf{L}^2(\mathbb{R})$ is defined as:

$$S_g f(u, \xi) = \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-i\xi t}\, dt$$

Here we write $S_g f$ rather than Sf to emphasize the dependence upon the window choice g. Up till now we have been a little sloppy in that we do not know if this transform is well defined. To that end, the next theorem is an analogue of Parseval's formula (Theorem 6) for the windowed Fourier transform and for windows g in a subclass of $\mathbf{L}^2(\mathbb{R})$. Besides showing that $S_g f$ is well defined for $f \in \mathbf{L}^2(\mathbb{R})$, like the original Parseval formula it is extremely useful.

Theorem 22. *Let $f, h \in \mathbf{L}^2(\mathbb{R})$ and let g be a real symmetric function with $g \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ and $\|g\|_2 = 1$. Then:*

$$\langle f, h \rangle = \frac{1}{2\pi} \langle S_g f, S_g h \rangle_{\mathbf{L}^2(\mathbb{R}^2)}$$

Proof. We will need another fundamental result from real analysis, which is Young's inequality. Suppose that $f_1 \in \mathbf{L}^p(\mathbb{R})$, $f_2 \in \mathbf{L}^q(\mathbb{R})$, and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

Then

$$\|f_1 * f_2\|_r \leq \|f_1\|_p \|f_2\|_q$$

Now define f_ξ as $f_\xi(u) = S_g f(u, \xi)$, so that we think of the windowed Fourier transform as a function in u with a parameter ξ. We first show that $f_\xi \in \mathbf{L}^2(\mathbb{R})$ and then compute its Fourier transform. Additionally, set

$g_\xi(t) = g(t)e^{i\xi t}$; we can rewrite $f_\xi(u)$ as (using that g is symmetric):

$$\begin{aligned}
f_\xi(u) &= \int_{-\infty}^{+\infty} f(t)g(t-u)e^{-i\xi t}\,dt \\
&= e^{-iu\xi}\int_{-\infty}^{+\infty} f(t)g(u-t)e^{i\xi(u-t)}\,dt \\
&= e^{-iu\xi} f * g_\xi(u)
\end{aligned}$$

It thus follows, using Young's inequality, that

$$\|f_\xi\|_2 = \|f * g_\xi\|_2 \leq \|g\|_1\|f\|_2$$

The Fourier transform of f_ξ is computed as:

$$\widehat{f_\xi}(\omega) = \widehat{f}(\omega+\xi)\widehat{g}_\xi(\omega+\xi) = \widehat{f}(\omega+\xi)\widehat{g}(\omega)$$

Let us now compute the inner product between $S_g f$ and $S_g h$. Since $f_\xi, h_\xi \in \mathbf{L}^2(\mathbb{R})$ we can use Parseval's formula and our computation for their Fourier transform to get:

$$\begin{aligned}
\frac{1}{2\pi}\langle S_g f, S_g h\rangle &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_g f(u,\xi) S_g h^*(u,\xi)\,du\,d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f_\xi(u)h_\xi^*(u)\,du\right)d\xi \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)|\widehat{g}(\omega)|^2\,d\omega\,d\xi \qquad (14)
\end{aligned}$$

We would like to switch the order of integration using Fubini. To do so we need to bound:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)|\widehat{g}(\omega)|^2\,d\omega\,d\xi \\
&\qquad = \int_{-\infty}^{+\infty}|\widehat{g}(\omega)|^2\int_{-\infty}^{+\infty}|\widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)|\,d\xi\,d\omega \\
&\qquad \leq \int_{-\infty}^{+\infty}|\widehat{g}(\omega)|^2\left(\int_{-\infty}^{+\infty}|\widehat{f}(\omega+\xi)|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}|\widehat{h}(\omega+\xi)|^2\,d\xi\right)^{\frac{1}{2}}d\omega \\
&\qquad \leq (2\pi)^2\|g\|_2^2\|f\|_2\|h\|_2 < \infty
\end{aligned}$$

Thus we can apply Fubini and continuing from (14) we have:

$$\begin{aligned}
(14) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{g}(\omega)|^2 \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(\omega+\xi)\widehat{h}^*(\omega+\xi)\,d\xi \right) d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{g}(\omega)|^2 \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{f}(\xi)\widehat{h}^*(\xi)\,d\xi \right) d\omega \\
&= \langle f, h\rangle \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{g}(\omega)|^2\,d\omega \\
&= \langle f, g\rangle
\end{aligned}$$

□

The windowed Fourier transform can be extended to any real, symmetric window $g \in \mathbf{L}^2(\mathbb{R})$ using a density argument. Using this extension, we can also extend Theorem 22 to any real symmetric window $g \in \mathbf{L}^2(\mathbb{R})$.

Corollary 23. *Let $f, h \in \mathbf{L}^2(\mathbb{R})$ and let g be a real symmetric function with $g \in \mathbf{L}^2(\mathbb{R})$ and $\|g\|_2 = 1$. Then:*

$$\langle f, h\rangle = \frac{1}{2\pi}\langle S_g f, S_g h\rangle_{\mathbf{L}^2(\mathbb{R}^2)}$$

It follows from Theorem 22 that $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$ and that it preserves the norm, up to a factor of $\sqrt{2\pi}$. This is the analog of the Plancheral formula; we collect it in the next corollary.

Corollary 24. *Let $g \in \mathbf{L}^2(\mathbb{R})$. The windowed Fourier transform is a linear map $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$, and it is also an isometry up to a factor of $\sqrt{2\pi}$:*

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|S_g f\|_{\mathbf{L}^2(\mathbb{R}^2)}$$

The windowed Fourier transform is highly redundant and covers the entire time frequency plane. Intuitively, we should have more than enough information to invert this transform. Indeed, that is the case:

Theorem 25. *Let g be a real symmetric window with $g \in \mathbf{L}^2(\mathbb{R})$ and $\|g\|_2 = 1$. Then for all $f \in \mathbf{L}^2(\mathbb{R})$,*

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} S_g f(u,\xi) g(t-u) e^{i\xi t}\,du\,d\xi$$

We are going to prove the inversion theorem using techniques from functional analysis. We collect the main points first. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$, and let $\ell : \mathcal{H} \to \mathbb{C}$ be a linear functional. We say that ℓ is continuous if for $v, h \in \mathcal{H}$ we have

$$\lim_{\|h\|\to 0} \|\ell(v+h) - \ell(v)\| = 0$$

The linear functional ℓ is bounded if there exists a universal constant $C \geq 0$ such that

$$|\ell(v)| \leq C\|v\|, \quad \forall\, v \in \mathcal{H}$$

It is a well known fact that linear functionals are continuous if and only if they are bounded.

Now let $\ell : \mathcal{H} \to \mathbb{C}$ be a continuous linear functional. The *Riesz Representation Theorem* states that for each such ℓ, there exists a unique $h \in \mathcal{H}$ such that

$$\ell(v) = \langle v, h\rangle, \quad \forall\, v \in \mathcal{H}$$

Since it is clear that the mappings $v \mapsto \langle v, h\rangle$ are continuous linear functionals for very $h \in \mathcal{H}$, the Riesz representation theorem shows that there is a bijective correspondence between $\mathcal{H}$ and continuous linear functionals on $\mathcal{H}$.

Finally, suppose now that $F : \mathbb{R} \to \mathcal{H}$, so that for each $u \in \mathbb{R}$, $F(u)$ is an element of the Hilbert space $\mathcal{H}$. One can think of F as a "vector valued function." For example, if $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$ then $F(u)(t)$ is a square integrable function in the variable $t \in \mathbb{R}$ for each $u \in \mathbb{R}$. Using F, one can define a linear functional

$$\ell_F(v) = \int_{-\infty}^{+\infty} \langle v, F(u)\rangle \, du$$

If ℓ_F is bounded / continuous, then by the Riesz representation theorem there exists a unique element $\widetilde{f} \in \mathcal{H}$ such that $\ell_F(v) = \langle v, \widetilde{f}\rangle$. Thus

$$\langle v, \widetilde{f}\rangle = \int_{-\infty}^{+\infty} \langle v, F(u)\rangle \, du, \quad \forall\, v \in \mathcal{H} \tag{15}$$

We write

$$\widetilde{f} = \int_{-\infty}^{+\infty} F(u)\, du$$

which means that (15) holds; this is a type of weak equality. We are going to prove Theorem 25 in this sense.

Proof of Theorem 25. Define a linear functional $\ell : \mathbf{L}^2(\mathbb{R}) \to \mathbb{C}$ as

$$\ell(h) = \frac{1}{2\pi} \langle S_g h, S_g f \rangle_{\mathbf{L}^2(\mathbb{R}^2)}, \quad \forall\, h \in \mathbf{L}^2(\mathbb{R})$$

By the Parseval theorem for windowed Fourier transforms (Theorem 22), we have:

$$\ell(h) = \langle h, f \rangle \leq \|f\|_2 \|h\|_2$$

Thus ℓ is a bounded, and hence continuous, linear functional. At this point we could apply the Riesz representation theorem, but it would just tell us what we already know which is that $\ell(h) = \langle h, f \rangle$. Instead we come up with a "vector valued function" $F(u,\xi) \in \mathbf{L}^2(\mathbb{R})$ and show that

$$\ell(h) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle h, F(u,\xi) \rangle \, du \, d\xi \tag{16}$$

To do so, recall that we defined

$$g_{u,\xi}(t) = g(t-u) e^{i\xi t}$$

and that we can write

$$Sf(u,\xi) = \langle f, g_{u,\xi} \rangle$$

We have:

$$\begin{aligned} \ell(h) = \frac{1}{2\pi} \langle S_g h, S_g f \rangle_{\mathbf{L}^2(\mathbb{R}^2)} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_g h(u,\xi) S_g f^*(u,\xi) \, du \, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle h, g_{u,\xi} \rangle S_g f^*(u,\xi) \, du \, d\xi \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \langle h, (2\pi)^{-1} S_g f(u,\xi) g_{u,\xi} \rangle \, du \, d\xi \end{aligned}$$

Therefore we have verified (16) with

$$F(u,\xi) = \frac{1}{2\pi} S_g f(u,\xi) g_{u,\xi}$$

It follows that $\ell(h)$ can be written as

$$\ell(h) = \langle h, \widetilde{f} \rangle$$

with
$$\widetilde{f} = \int_{-\infty}^{+\infty} F(u,\xi)\,du\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_g f(u,\xi) g_{u,\xi}$$
But then $\ell(h) = \langle h, f\rangle = \langle h, \widetilde{f}\rangle$ for all $h \in \mathbf{L}^2(\mathbb{R})$, and so $f = \widetilde{f}$ (in the weak sense) and the inversion formula is proved. $\square$

Exercise 29. Read Section 4.2.1 of *A Wavelet Tour of Signal Processing.*

4.2.2 Choice of the Window

The time frequency localization of the window g can be modified with a scaling. Suppose that the Heisenberg boxes of the time frequency atoms $g_{u,\xi}$ have time width σ_t and frequency width σ_ω. Let
$$g_s(t) = s^{-1/2} g(s^{-1}t)$$
be a dilation of g by the time scale s. One can show that if we replace the window g with g_s, then the resulting Heisenberg box has time width $s\sigma_t$ and frequency width $s^{-1}\sigma_\omega$. While the area remains $\sigma_t\sigma_\omega$, the resolution in time is modified by s while the resolution in frequency is modified by s^{-1}. Depending on the signal type we may want better localization in time or frequency, or a balance of both; the parameter s allows us to adjust accordingly while keeping the time frequency area of each box constant.

In numerical applications, the localized waveforms $g_{u,\xi}(t)$ can only be sampled a finite number of times, which means the support of the window g must be compact or it must be restricted to a compact set (as in the case of a Gaussian window). If g has compact support, then $\widehat{g}$ must have an infinite support. Since g is symmetric and often $g(t) \geq 0$ for all t, $\widehat{g}(\omega)$ will be symmetric with a main "lobe" (bump) centered at $\omega = 0$, which decays to zero with oscillations; see Figure 10.

The frequency resolution of the windowed Fourier transform is determined by the spread of $\widehat{g}$ around $\omega = 0$. Previously we used σ_ω to measure this spread, however the following three parameters give a more fine grained measure:

- The bandwidth $\Delta\omega$, which is defined by:
$$\frac{|\widehat{g}(\Delta\omega/2)|^2}{|\widehat{g}(0)|^2} = \frac{1}{2}$$
This measures the energy concentration of $\widehat{g}_{u,\xi}(\omega)$ around $\omega = \xi$.

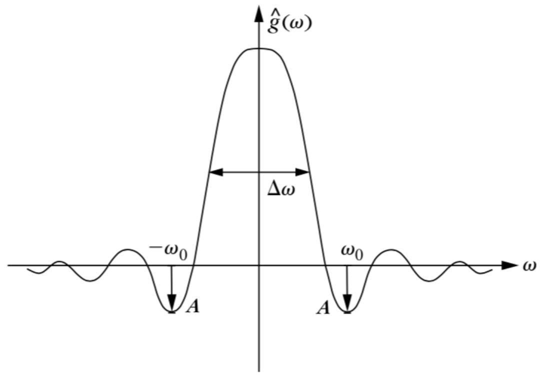

Figure 10: The energy spread of $\widehat{g}(\omega)$ is measured by its bandwidth and the maximum amplitude A of the first side lobes, located at $\pm\omega_0$.

- The maximum amplitude A of the first side lobes located at $\omega = \pm\omega_0$. The important thing is the side lobe amplitude relative to the amplitude of the central lobe at $\omega = 0$; this ratio can be measured in decibels:

$$A = 10 \log_{10} \frac{|\widehat{g}(\omega_0)|^2}{|\widehat{g}(0)|^2}$$

  Side lobes create echos of the response $Sf(u, \xi)$ at $Sf(u, \xi \pm \omega_0)$. If A is small (i.e., very negative), then the side lobe magnitude is small relative the main lobe amplitude and these echos will be negligible relative to the response at ξ.

- The polynomial exponent p, which gives the asymptotic decay of $|\widehat{g}(\omega)|$ for large frequencies,

$$|\widehat{g}(\omega)| = O(|\omega|^{-(p+1)})$$

  This is important of several localized frequency phenomena occur close together in the time frequency plane. In this case it can be hard to "unmix" the various frequency tones unless p is large. We obtain a large p by using a smooth window.

Exercise 30. Read Sections 4.2.2 and 4.2.3 of *A Wavelet Tour of Signal Processing*.

4.3 Time Frequency Geometry of Instantaneous Frequencies

When listening to music we perceive several frequencies that change with time. This leads to the notion of an *instantaneous frequency*, which we define here at the outset.

4.3.1 Instantaneous Frequency

Section 4.4.1 of A Wavelet Tour of Signal Processing

If $f : \mathbb{R} \to \mathbb{C}$ is complex valued then $f(t)$ can be uniquely represented as

$$f(t) = a(t)e^{i\theta(t)}$$

where $a(t) = |f(t)|$ is the amplitude of $f(t)$ and $\theta(t) \in [0, 2\pi)$ is the phase of $f(t)$. In this case, we define the *instantaneous frequency* of $f(t)$ as $\theta'(t)$.

For real valued signals $f : \mathbb{R} \to \mathbb{R}$, we would like to decompose $f(t)$ as

$$f(t) = \alpha(t) \cos \vartheta(t)$$

However, this representation is not unique since it has two parameters $\alpha(t)$ and $\vartheta(t)$ for each real value $f(t)$. We settle on a particular representation by defining the analytic part of $f(t)$.

A function $h_a \in \mathbf{L}^2(\mathbb{R})$ is *analytic* if

$$\widehat{h}_a(\omega) = 0, \quad \forall\, \omega < 0$$

An analytic function is necessarily complex valued but is entirely characterized by its real part. Indeed, define $h(t) = \Re[h_a(t)]$ to be the real part of $h_a(t)$. Its Fourier transform is:

$$\widehat{h}(\omega) = \frac{\widehat{h}_a(\omega) + \widehat{h}_a^*(-\omega)}{2}$$

which in turn yields:

$$\widehat{h}_a(\omega) = \begin{cases} 2\widehat{h}(\omega) & \omega > 0 \\ \widehat{h}(\omega) & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

If we start with a real valued signal $f(t)$ then we define the analytic part $f_a(t)$ of $f(t)$ as the inverse Fourier transform of

$$\widehat{f_a}(\omega) = \begin{cases} 2\widehat{f}(\omega) & \omega > 0 \\ \widehat{f}(\omega) & \omega = 0 \\ 0 & \omega < 0 \end{cases}$$

Since the analytic part $f_a(t)$ of $f(t)$ is complex valued, it can be decomposed uniquely as

$$f_a(t) = a(t)e^{i\theta(t)}$$

Since $f(t) = \Re[f_a(t)]$ we have that

$$f(t) = a(t)\cos\theta(t)$$

This representation is uniquely defined because it is derived from the analytic part of f. We call $a(t)$ the analytic amplitude of $f(t)$ and $\theta'(t)$ its instantaneous frequency.

As a somewhat obvious example we compute the analytic part of the real valued signal

$$f(t) = a\cos(\omega_0 t + \theta_0) = \frac{a}{2}\left(e^{i(\omega_0 t+\theta_0)} + e^{-i(\omega_0 t+\theta_0)}\right)$$

Its Fourier transform is:

$$\widehat{f}(\omega) = \pi a\left(e^{i\theta_0}\delta(\omega-\omega_0) + e^{-i\theta_0}\delta(\omega+\omega_0)\right)$$

If $\omega_0 > 0$, then the Fourier transform of the analytic part is:

$$\widehat{f_a}(\omega) = 2\widehat{f}(\omega) = 2\pi a e^{i\theta_0}\delta(\omega-\omega_0), \quad \omega \geq 0$$

and thus

$$f_a(t) = ae^{i(\omega_0 t+\theta_0)}$$

If we replace the constant a with an amplitude function $a(t)$, so that

$$f(t) = a(t)\cos(\omega_0 t + \theta_0)$$

then the Fourier transform of $f(t)$ is:

$$\widehat{f}(\omega) = \frac{1}{2}\left(e^{i\theta_0}\widehat{a}(\omega-\omega_0) + e^{-i\theta_0}\widehat{a}(\omega+\omega_0)\right)$$

If the variations of $a(t)$ are slow compared to the period $2\pi/\omega_0$, then it must be that $\operatorname{supp}\widehat{a} \subseteq [-\omega_0, \omega_0]$. In this case:

$$\widehat{f_a}(\omega) = 2\widehat{f}(\omega) = e^{i\theta_0}\widehat{a}(\omega - \omega_0), \quad \omega \geq 0$$

and

$$f_a(t) = a(t)e^{i(\omega_0 t + \theta_0)}$$

Let us now consider a slightly more complicated example:

$$f(t) = a\cos(\omega_1 t) + a\cos(\omega_2 t)$$

In this case the analytic part of the signal is given by:

$$\begin{aligned} f_a(t) &= ae^{i\omega_1 t} + ae^{i\omega_2 t} \\ &= 2a\cos\left(\frac{(\omega_1 - \omega_2)t}{2}\right) e^{i(\omega_1+\omega_2)t/2} \end{aligned}$$

Thus the instantaneous frequency is

$$\theta'(t) = \frac{\omega_1 + \omega_2}{2}$$

and the amplitude is

$$a(t) = 2a\left|\cos\left(\frac{(\omega_1 - \omega_2)t}{2}\right)\right|$$

The result is unsatisfying because the instantaneous frequency is the average of the frequencies of the two cosine waves. We would have no indication (forgetting the amplitude) that the signal is not in fact one cosine with frequency $(\omega_1 + \omega_2)/2$, but rather two separate cosines.

More generally, one would like to be able to analyze signals of the form

$$f(t) = \sum_{k=1}^{K} a_k(t)\cos\theta_k(t) \tag{17}$$

where $a_k(t)$ and $\theta_k(t)$ vary slowly in time. Such decompositions can be used to model music and other auditory signals. We want to isolate the different amplitudes $a_k(t)$ and instantaneous frequencies $\theta_k'(t)$. A windowed Fourier transform can help with this.

Exercise 31. Read Section 4.4.1 of *A Wavelet Tour of Signal Processing.*

4.3.2 Windowed Fourier Ridges

Section 4.4.2 of A Wavelet Tour of Signal Processing.

We are going to use the windowed Fourier transform, and in particular the local maxima of the windowed Fourier transform, to isolate individual amplitudes $a_k(t)$ and instantaneous frequencies $\theta_k'(t)$ as in the signal model (17).

We make some additional assumptions on the real symmetric window $g(t)$. We suppose that:

- $\operatorname{supp} g = [-1/2, 1/2]$
- $g(t) \geq 0$ so that $|\widehat{g}(\omega)| \leq \widehat{g}(0)$ for all $\omega \in \mathbb{R}$
- $\|g\|_2 = 1$ but also $\widehat{g}(0) = \int g(t)\, dt = \|g\|_1 \approx 1$

For a scale σ set

$$g_\sigma(t) = \sigma^{-1/2} g(\sigma^{-1} t)$$

Note that

$$\operatorname{supp} g_\sigma = [-\sigma/2, \sigma/2] \quad \text{and} \quad \|g_\sigma\|_2 = 1$$

We define the windowed Fourier transform with the scale parameter σ as:

$$S_\sigma f(u, \xi) = \int_{-\infty}^{+\infty} f(t) g_\sigma(t-u) e^{-i\xi t}\, dt$$

The next theorem relates $S_\sigma f(u,\xi)$ to the instantaneous frequency of $f(t)$.

Theorem 26. *Let $f(t) = a(t)\cos\theta(t)$. If $\xi \geq 0$, then*

$$S_\sigma f(u,\xi) = \frac{\sqrt{\sigma}}{2} a(u) e^{i[\theta(u) - \xi u]} \Big(\widehat{g}(\sigma[\xi - \theta'(u)]) + \varepsilon(u,\xi) \Big)$$

where

$$|\varepsilon(u,\xi)| \leq \varepsilon_{a,1}(u,\xi) + \varepsilon_{a,2}(u,\xi) + \varepsilon_{\theta,2}(u,\xi) + \sup_{|\omega| \geq \sigma\theta'(u)} |\widehat{g}(\omega)|$$

with

$$\varepsilon_{a,1}(u,\xi) \leq \frac{\sigma |a'(u)|}{|a(u)|}$$

and

$$\varepsilon_{a,2}(u,\xi) \leq \sup_{|t-u| \leq \sigma/2} \frac{\sigma^2 |a''(t)|}{|a(u)|}$$

Furthermore, if $\sigma|a'(u)||a(u)|^{-1} \leq 1$*, then*

$$\varepsilon_{\theta,2}(u,\xi) \leq \sup_{|t-u|\leq\sigma/2} \sigma^2|\theta''(t)|$$

And finally, if $\xi = \theta'(u)$*, then*

$$\varepsilon_{a,1}(u,\xi) = \frac{\sigma|a'(u)|}{|a(u)|}|\widehat{g}'(2\sigma\theta'(u))|$$

We omit the proof, which is given in pages 119–122 of *A Wavelet Tour of Signal Processing*. If we can neglect the error term $\varepsilon(u,\xi)$, then we will see that $S_\sigma f(u,\xi)$ enables us to measure $a(u)$ and $\theta'(u)$. This will be the case if $a(t)$ and $\theta(t)$ vary slowly. In particular, $\varepsilon_{a,1}$ is small if $a(t)$ varies slowly over the whole real line, while $\varepsilon_{a,2}$ and $\varepsilon_{\theta,2}$ only require the second derivatives of $a(t)$ and $\theta(t)$ to be small over an interval of length equal to the support of the window g. The fourth part of the error term is small if

$$\Delta\omega \leq \sigma\theta'(u) \tag{18}$$

where recall $\Delta\omega$ is the bandwidth of g.

Exercise 32. Theorem 26 is Theorem 4.6 (p. 119) in *A Wavelet Tour of Signal Processing*. Since we will not prove this theorem in class, read its proof in preparation for the next class.

Exercise 33. Corollary 23 leaves out some details which we are going to fill in now. First recall that we proved that $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$ whenever $g \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ and g is real and symmetric. Prove that for any real, symmetric $g \in \mathbf{L}^2(\mathbb{R})$, we can also define a windowed Fourier transform $S_g : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R}^2)$ using a density argument similar to the one we used to extend the Fourier transform to $\mathbf{L}^2(\mathbb{R})$. Now prove the windowed Fourier Parseval formula (Corollary 23).

Exercise 34. The analytic part $x_a[n]$ of a real valued discrete signal $x \in \mathbb{R}^N$ is defined by

$$\widehat{x}_a[n] = \begin{cases} \widehat{x}[k] & k = 0, N/2 \\ 2\widehat{x}[k] & 0 < k < N/2 \\ 0 & N/2k < N \end{cases}$$

(a) Suppose that $y \in \mathbb{C}^N$ is a complex valued discrete signal and let $y_r[n] = \Re(y[n])$ be the real part of y. Prove that

$$\widehat{y}_r[k] = \frac{\widehat{y}[k] + \widehat{y}^*[-k]}{2}$$

(b) For $x \in \mathbb{R}^N$ prove that $\Re(x_a) = x$.

Exercise 35. We are going to use your windowed Fourier transform code to reproduce some results from the book.

(a) Read Example 4.5 (p. 94) of *A Wavelet Tour of Signal Processing* and determine what the signal is (write it out analytically). Then compute the windowed Fourier transform and corresponding spectrogram, and recreate something similar to Figure 4.3(a). Provide a plot of your spectrogram.

(b) Consider the signal

$$f(t) = a_1 \cos(bt^2 + ct) + a_2 \cos(bt^2)$$

which consists of two real valued linear chirps. Compute the windowed Fourier transform and spectrogram of $f(t)$. Can you find a window g and parameters a_1, a_2, b, c such that you can recreate something similar to Figure 4.13(a)? Provide a plot of your spectrogram.

(c) Consider the signal

$$f(t) = a_1 \cos\left(\frac{\alpha_1}{\beta_1 - t}\right) + a_2 \cos\left(\frac{\alpha_2}{\beta_2 - t}\right)$$

which consists of two hyperbolic chirps. Select parameters $a_1, a_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ and compute the windowed Fourier transform and spectrogram of $f(t)$. Do you get something like Figure 4.14(a)? Provide a plot of your spectrogram.

Exercise 36. Let $f(t) = e^{i\theta(t)}$ and let g be a real, symmetric window function with $\|g\|_2 = 1$.

(a) Prove that

$$\int_{-\infty}^{+\infty} |Sf(u,\xi)|^2 \, d\xi = 2\pi, \quad \forall\, u \in \mathbb{R}$$

(b) Prove that

$$\int_{-\infty}^{+\infty} \xi |Sf(u,\xi)|^2 \, d\xi = 2\pi \int_{-\infty}^{+\infty} \theta'(t) |g(t-u)|^2 \, dt, \quad \forall\, u \in \mathbb{R}$$

and interpret this result.

References

[1] Stéphane Mallat. *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way.* Academic Press, 3rd edition, 2008.

[2] Yves Meyer. *Wavelets and Operators*, volume 1. Cambridge University Press, 1993.

[3] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction.* Princeton Lectures in Analysis. Princeton University Press, 2003.

[4] Richard L. Wheeden and Antoni Zygmund. *Measure and Integral: An Introduction to Real Analysis.* Marcel Dekker, 1977.

[5] Elias M. Stein and Guido Weiss. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, 1971.

[6] John J. Benedetto and Matthew Dellatorre. Uncertainty principles and weighted norm inequalities. *Contemporary Mathematics*, 693:55–78, 2017.

[7] Karlheinz Gröchenig. *Foundations of Time Frequency Analysis.* Springer Birkhäuser, 2001.