We can use this link with Taylor’s theorem to define notions of local regularity rather than global regularity. In particular, a function \( f(t) \) is pointwise Lipschitz \( \alpha \geq 0 \) at \( v \in \mathbb{R} \) if there exists \( K > 0 \) and a polynomial \( p_v(t) \) of degree \( n = \lfloor \alpha \rfloor \) such that

\[
|f(t) - p_v(t)| \leq K|t - v|^{\alpha}
\]  

(30)

Furthermore, a function \( f \) is uniformly Lipschitz over an interval \([a, b]\) if it satisfies (30) for all \( v \in [a, b] \) with a constant \( K \) that is independent of \( v \).

At each \( v \in \mathbb{R} \) the polynomial \( p_v(t) \) is unique. Additionally, if \( f \) is \( n = \lfloor \alpha \rfloor \) times continuously differentiable in a neighborhood of \( v \), then \( p_v(t) = J_v f(t) \).

A function that is bounded but discontinuous at \( v \) is Lipschitz 0 at \( v \). If \( \alpha < 1 \) at \( v \), then \( f \) is not differentiable at \( v \) and \( \alpha \) characterizes the type of singularity.

The next theorem generalizes Theorem 11 by relating the decay of the Fourier transform of \( f(t) \) to the \( \alpha \) regularity of \( f \).

**Theorem 29.** Suppose that \( f \in L^1(\mathbb{R}) \). If

\[
\int_{-\infty}^{+\infty} |\hat{f}(\omega)|(1 + |\omega|^{\alpha}) \, d\omega < +\infty
\]

(31)

then \( f \in C^{\alpha}(\mathbb{R}) \).

**Proof.** Equation (31) implies that \( \hat{f} \in L^1(\mathbb{R}) \), and so the Fourier inversion formula (2) holds. We use it to prove \( f \in L^\infty(\mathbb{R}) \):

\[
|f(t)| \leq \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t} \, d\omega \right| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)| \, d\omega \\
\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|(1 + |\omega|^{\alpha}) \, d\omega < \infty
\]

Now suppose that \( 0 \leq \alpha < 1 \) and show that \( f \in \dot{C}^{\alpha}(\mathbb{R}) \). To do so we need to show there exists \( K > 0 \) such that

\[
|f(t) - f(v)| \leq K|t - v|^{\alpha}, \quad \forall t, v \in \mathbb{R}
\]

By the Fourier inversion formula (2) we have that

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)e^{i\omega t} \, d\omega
\]
It follows that
\[ |f(t) - f(v)| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)||e^{i\omega t} - e^{i\omega v}| \frac{1}{|t - v|^\alpha} d\omega \]

For $|\omega| \geq |t - v|^{-1}$,
\[ \frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^\alpha} \leq \frac{2}{|t - v|^\alpha} \leq 2|\omega|^\alpha \quad (32) \]

On the other hand, for $|\omega| \leq |t - v|^{-1}$, we note that if a function $h \in \dot{C}^1(\mathbb{R})$, then
\[ |h(t) - h(v)| \leq K|t - v|, \quad K = \sup_{u \in \mathbb{R}} |h'(u)| \]

Note that $e_\omega \in C^1(\mathbb{R})$, where $e_\omega(t) = e^{i\omega t}$, and $|e'_\omega(t)| = |\omega|$. Therefore,
\[ \frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^\alpha} \leq \frac{|\omega||t - v|}{|t - v|^\alpha} = |\omega||t - v|^{1-\alpha} \leq |\omega||\omega|^{\alpha-1} = |\omega|^\alpha \quad (33) \]

Combining (32) and (33), we obtain
\[ \frac{|f(t) - f(v)|}{|t - v|^\alpha} \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2|\hat{f}(\omega)||\omega|^\alpha d\omega = K \]

Equation (31) ensures that $K < \infty$, and so $f \in C^\alpha(\mathbb{R})$.

We now extend the result to $\alpha \geq 1$. Let $n = \lfloor \alpha \rfloor$. Theorem 11 proves that $f \in C^n(\mathbb{R})$. Recall that $\widehat{f^{(n)}}(\omega) = (i\omega)^n \hat{f}(\omega)$. Equation (31) gives:
\[ \int_{-\infty}^{+\infty} |\widehat{f^{(n)}}(\omega)|(1 + |\omega|^{\alpha-n}) d\omega = \int_{-\infty}^{+\infty} |\hat{f}(\omega)|(|\omega|^n + |\omega|^\alpha) d\omega < \infty \]

Thus by our work above, we have that $f^{(n)} \in C^{\alpha-n}(\mathbb{R})$, which proves that $f \in C^\alpha(\mathbb{R})$.

As we have discussed previously for $C^n$-smooth functions, the decay of the Fourier transform can only indicate the minimum regularity of $f(t)$. Wavelet transforms characterize both the global and pointwise regularity of functions.

**Exercise 50.** Read Section 6.1.1 of *A Wavelet Tour of Signal Processing.*
Exercise 51. Consider the function

\[ f(t) = t \sin \left( \frac{1}{t} \right) \]

(a) [5 points] Prove that \( f(t) \) is pointwise Lipschitz 1 for all \( t \in (-1, 1) \).

(b) [15 points] Prove that \( f \in C^\alpha(-1, 1) \) only for \( \alpha \leq 1/2 \) (Hint: Consider the points \( t_n = (n + 1/2)^{-1}\pi^{-1} \)).

Exercise 52. Show that \( f \) may be pointwise Lipschitz \( \alpha > 1 \) at \( v \), while \( f' \) is not pointwise Lipschitz \( \alpha - 1 \) at \( v \). Consider \( f(t) = t^2 \cos(1/t) \) at \( t = 0 \).

Exercise 53. Find a function \( f(t) \) for which there exists a constant \( K \) such that

\[ |f(t) - f(v)| \leq K|u - v|, \quad \forall t, v \in \mathbb{R} \]

but does not satisfy (31) with \( \alpha = 1 \).

5.1.2 Wavelet Vanishing Moments

Section 6.1.2 of A Wavelet Tour of Signal Processing.

We assume throughout that \( \psi(t) \) is a real valued wavelet.

A wavelet \( \psi \) has \( n \) vanishing moments if

\[ \int_{-\infty}^{+\infty} t^k \psi(t) \, dt = 0, \quad \forall 0 \leq k < n \]

A wavelet \( \psi \) with \( n \) vanishing moments is orthogonal to polynomials of degree \( n - 1 \).

Suppose now that \( f \) is Lipschitz \( \alpha < n \) at \( v \), so that

\[ f(t) = p_v(t) + \varepsilon_v(t) \]

with \( p_v(t) \) a polynomial of degree \( n - 1 \) and

\[ |\varepsilon_v(t)| \leq K|t - v|^\alpha \]

We have that

\[ Wp_v(u, s) = \int_{-\infty}^{+\infty} p_v(t) \frac{1}{\sqrt{s}} \psi \left( \frac{t - u}{s} \right) \, dt = \sqrt{s} \int_{-\infty}^{+\infty} p_v(st' + u)\psi(t') \, dt' = 0 \]
Therefore,

$$Wf(u, s) = Wp_v(u, s) + W\varepsilon_v(u, s) = W\varepsilon_v(u, s)$$

Thus a wavelet transform with $n$ vanishing moments analyzes $f(t)$ around $t = v$ by ignoring the polynomial approximation of $f(t)$ and focusing on the residual $\varepsilon_v(t)$.

A wavelet $\psi$ has fast decay if

$$\forall m \in \mathbb{N}, \exists C_m \text{ such that } |\psi(t)| \leq \frac{C_m}{1 + |t|^m}, \forall t \in \mathbb{R}$$

The following theorem shows that a wavelet $\psi$ with fast decay and $n$ vanishing moments is the $n^{th}$ derivative of a function $\theta(t)$. The resulting wavelet transform is thus a multiscale differential operator.

**Theorem 30.** A wavelet $\psi(t)$ with fast decay has $n$ vanishing moments if and only if there exists $\theta(t)$ with a fast decay such that

$$\psi(t) = (-1)^n \theta^{(n)}(t)$$

Consequently,

$$Wf(u, s) = s^n \frac{d^n}{du^n}(f \ast \tilde{\theta}_s)(u)$$

where $\tilde{\theta}_s(t) = s^{-1/2} \theta(-t/s)$. Furthermore, $\psi$ has no more than $n$ vanishing moments if and only if

$$\int_{-\infty}^{+\infty} \theta(t) \, dt \neq 0$$

**Proof.** Suppose that $\psi$ has fast decay and $n$ vanishing moments. Since $\psi$ has fast decay we must have that $\hat{\psi} \in C^\infty(\mathbb{R})$; this follows from Theorem 11 by setting $f = \hat{\psi}$. Thus we can differentiate $\hat{\psi}(\omega)$ as many times as we like.

Recall that the Fourier transform of $h(t) = (-it)^k \psi(t)$ is $\hat{h}(\omega) = \hat{\psi}^{(k)}(\omega)$. It follows that

$$\hat{\psi}^{(k)}(0) = \int_{-\infty}^{+\infty} (-it)^k \psi(t) \, dt = (-i)^k \int_{-\infty}^{+\infty} t^k \psi(t) \, dt = 0, \forall 0 \leq k < n$$

We can therefore write $\hat{\psi}$ as

$$\hat{\psi}(\omega) = (-i\omega)^n \hat{\theta}(\omega)$$
where \( \hat{\theta} \in L^\infty \) since \( \hat{\psi} \in L^\infty(\mathbb{R}) \). It follows that
\[
\psi(t) = (-1)^n \theta^{(n)}(t)
\]

The fast decay of \( \theta(t) \) is proved with an induction on \( n \). For \( n = 1 \),
\[
\hat{\psi}(\omega) = -i\omega \hat{\theta}(\omega) \implies \psi(t) = -\theta'(t)
\]
It follows that
\[
\theta(t) = - \int_{-\infty}^{t} \psi(u) \, du
\]
Thus, using the fast decay of \( \psi(t) \),
\[
|\theta(t)| \leq \int_{-\infty}^{t} |\psi(u)| \, du \leq \int_{-\infty}^{t} \frac{C_m}{1 + |u|^m} \, du \leq \frac{C'_m}{1 + |t|^{m-1}}, \quad \forall \ m \geq 2
\]
Now make the inductive hypothesis that if \( \Psi(t) \) is any wavelet with fast decay and
\[
\hat{\Psi}(\omega) = (-i\omega)^k \hat{\Theta}(\omega), \quad 1 \leq k \leq n
\]
then \( \Theta(t) \) has fast decay. Consider now a wavelet \( \psi \) with fast decay that has \( n + 1 \) vanishing moments, so that \( \hat{\psi}(\omega) = (-i\omega)^{n+1} \hat{\Theta}(\omega) \). Define
\[
\hat{\Theta}(\omega) = -i\omega \hat{\theta}(\omega) \implies \hat{\psi}(\omega) = (-i\omega)^n \hat{\Theta}(\omega)
\]
By the inductive hypothesis, \( \Theta(t) \) has fast decay. But then since \( \hat{\Theta}(\omega) = -i\omega \hat{\theta}(\omega) \), we can apply the inductive hypothesis again to conclude that \( \theta(t) \) has fast decay.

Conversely, suppose that \( \psi(t) = (-1)^n \theta^{(n)}(t) \) and \( \theta(t) \) has fast decay. Because of the fast decay,
\[
|\hat{\theta}(\omega)| \leq \int_{-\infty}^{+\infty} |\theta(t)| \, dt \leq \int_{-\infty}^{+\infty} \frac{C_m}{1 + |t|^m} \, dt < +\infty, \quad m \geq 2
\]
Thus \( \hat{\theta} \in L^\infty(\mathbb{R}) \). The Fourier transform of \( \psi(t) \) is
\[
\hat{\psi}(\omega) = (-i\omega)^n \hat{\Theta}(\omega)
\]
It follows that \( \hat{\psi}^{(k)}(0) = 0 \) for \( k < n \). But then
\[
\int_{-\infty}^{+\infty} t^k \psi(t) \, dt = i^k \hat{\psi}^{(k)}(0) = 0, \quad 0 \leq k < n
\]
Thus \( \psi(t) \) has \( n \) vanishing moments.

To test whether \( \psi(t) \) has more than \( n \) vanishing moments, we compute:

\[
\int_{-\infty}^{+\infty} t^n \psi(t) \, dt = i^n \hat{\psi}^{(n)}(0) = (-i)^n n! \hat{\theta}(0)
\]

Clearly then \( \psi \) has no more than \( n \) vanishing moments if and only if

\[
\hat{\theta}(0) = \int_{-\infty}^{+\infty} \theta(t) \, dt \neq 0
\]

Recall the wavelet transform can be written as

\[
Wf(u, s) = f * \widetilde{\psi}_s(u)
\]

where

\[
\widetilde{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi \left( \frac{-t}{s} \right) = \frac{(-1)^n}{\sqrt{s}} \theta^{(n)} \left( \frac{-t}{s} \right) = (-1)^n \widetilde{\theta}_s^{(n)}(t)
\]

A simple calculation also shows that

\[
\frac{d^n}{dt^n} \widetilde{\theta}_s(t) = \frac{1}{s^n} \left( \frac{-1}{\sqrt{s}} \right)^n \theta^{(n)} \left( \frac{-t}{s} \right) = \frac{(-1)^n}{s^n} \widetilde{\theta}_s^{(n)}(t) = \frac{\widetilde{\psi}_s(t)}{s^n}
\]

Therefore \( \widetilde{\psi}_s(t) = s^n (d^n / dt^n) \widetilde{\theta}_s(t) \). We then have:

\[
Wf(u, s) = f * \widetilde{\psi}_s(u) = s^n f * \theta_s^{(n)}(u) = s^n \frac{d^n}{du^n} (f * \theta)(u)
\]

\( \square \)

If \( K = \hat{\theta}(0) \neq 0 \), then the convolution \( f * \widetilde{\theta}_s(t) \) can be interpreted as a weighted average of \( f \) with a kernel dilated by \( s \). Theorem 30 proves that \( Wf(u, s) \) is an \( n \)th order derivative of an averaging of \( f \) over a domain proportional to \( s \) and centered at \( u \). Figure plots \( Wf(u, s) \) calculated with \( \psi(t) = -\theta'(t) \), where \( \theta(t) \) is a Gaussian. Notice how the sign and magnitude of the wavelet coefficients corresponds to the derivative of \( f \) averaged over a window of size proportional to \( s \). Compare to Figure 19, which computed \( Wf(u, s) \) with the Mexican hat wavelet \( \psi(t) = \theta''(t) \) (\( \theta \) again a Gaussian).

Since \( \theta(t) \) has fast decay, once can verify that for any \( f \) that is continuous at \( u \),

\[
\lim_{s \to 0} f * \frac{1}{\sqrt{s}} \theta_s(u) = Kf(u)
\]

92
Figure 22: Wavelet transform $W_f(u, s)$ calculated with $\psi = -\theta'$, where $\theta$ is a Gaussian, for the signal $f(t)$ shown in (a). Position parameter $u$ and scale $s$ vary, respectively, along the horizontal and vertical axes. (b) Black, gray, and white points correspond to positive, zero, and negative wavelet coefficients. Singularities create large-amplitude coefficients in their cone of influence.
In the sense of distributions, we write

$$\lim_{s \to 0} \frac{1}{\sqrt{s}} \tilde{\theta}_s(t) = K\delta(t)$$

If $f$ is $n$ times continuously differentiable in the neighborhood of $u$, then using Theorem 30,

$$\lim_{s \to 0} \frac{Wf(u, s)}{s^{n+1/2}} = \lim_{s \to 0} \frac{1}{\sqrt{s}} d^n(f \ast \tilde{\theta}_s)(u) = \lim_{s \to 0} f^{(n)}(u) \ast \frac{1}{\sqrt{s}} \tilde{\theta}_s(u) = Kf^{(n)}(u) \quad (34)$$

In particular, if $f \in C^n(\mathbb{R})$, then $|Wf(u, s)| = O(s^{n+1/2})$. This gives us a first relation between the decay of $|Wf(u, s)|$ as $s \to 0$ and the uniform regularity of $f$. Next we push harder and obtain finer relations.

**Exercise 54.** Read Section 6.1.2 of *A Wavelet Tour of Signal Processing*.

**Exercise 55.** Let $f(t) = |t|^\alpha$. Show that $Wf(u, s) = s^{\alpha+1/2}Wf(u/s, 1)$. Prove that it is not sufficient to measure the decay of $|Wf(u, s)|$ when $s \to 0$ at $u = 0$ in order to compute the Lipschitz regularity of $f$ at $t = 0$.

### 5.1.3 Regularity Measurements with Wavelets

*Section 6.1.3 of A Wavelet Tour of Signal Processing.*

We now prove that “zooming in” on the wavelet coefficients of a signal $f$ characterizes the pointwise regularity of $f$. We utilize a real valued wavelet $\psi \in C^n(\mathbb{R})$ with $n$ vanishing moments and with derivatives that have fast decay. The latter point means that for any $0 \leq k \leq n$ and $m \in \mathbb{N}$ there exists $C_m \geq 0$ such that

$$|\psi^{(k)}(t)| \leq \frac{C_m}{1 + |t|^m}, \quad \forall t \in \mathbb{R}$$

**Theorem 31.** If $f \in L^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ at $v \in \mathbb{R}$, then there exists $A > 0$ such that

$$|Wf(u, s)| \leq As^{\alpha+1/2} \left(1 + \left|\frac{u - v}{s}\right|^\alpha\right), \quad \forall (u, s) \in \mathbb{R} \times (0, \infty) \quad (35)$$

Conversely if $\alpha < n$ is not an integer and there exist $A$ and $\alpha' < \alpha$ such that

$$|Wf(u, s)| \leq As^{\alpha+1/2} \left(1 + \left|\frac{u - v}{s}\right|^{\alpha'}\right) \quad (36)$$
then \( f(t) \) is Lipschitz \( \alpha \) at \( t = v \).

Proof. We first prove (35). Since \( f \) is Lipschitz \( \alpha \) at \( v \), there exists a polynomial \( p_v(t) \) of degree \( |\alpha| < n \) and \( K \) such that

\[
|f(t) - p_v(t)| \leq K|t - v|^\alpha
\]

Recall that since \( \psi \) has \( n \) vanishing moments, \( W_{p_v}(u, s) = 0 \). Therefore:

\[
|Wf(u, s)| = \left| \int_{-\infty}^{+\infty} \left[ f(t) - p_v(t) \right] \frac{1}{\sqrt{s}} \psi \left( \frac{t - u}{s} \right) \, dt \right|
\]

\[
\leq \int_{-\infty}^{+\infty} K|t - v|^\alpha \frac{1}{\sqrt{s}} \left| \psi \left( \frac{t - u}{s} \right) \right| \, dt
\]

Now make the change of variables \( x = (t - u)/s \), which induces the change of measure \( dt = sdx \),

\[
|Wf(u, s)| \leq K \sqrt{s} \int_{-\infty}^{+\infty} |sx + u - v|^\alpha |\psi(x)| \, dx
\]

\[
\leq K \sqrt{s} \int_{-\infty}^{+\infty} 2^\alpha (|sx|^\alpha + |u - v|^\alpha) |\psi(x)| \, dx
\]

\[
= K2^\alpha s^{\alpha+1/2} \left( \int_{-\infty}^{+\infty} |x|^\alpha |\psi(x)| \, dx + \left| \frac{u - v}{s} \right|^\alpha \int_{-\infty}^{+\infty} |\psi(x)| \, dx \right)
\]

\[
\leq As^{\alpha+1/2} \left( 1 + \left| \frac{u - v}{s} \right|^\alpha \right)
\]

where

\[
A = K2^\alpha \max \left( \int_{-\infty}^{+\infty} |x|^\alpha |\psi(x)| \, dx, \|\psi\|_1 \right)
\]

and where we used the fact that \( |a + b|^\alpha \leq 2^\alpha (|a|^\alpha + |b|^\alpha) \).

Now we prove (36). This is a difficult proof that adapts the Littlewood-Paley approach referenced earlier in the course. Recall the real wavelet inverse formula from Theorem 28,

\[
f(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} Wf(u, s) \frac{1}{\sqrt{s}} \psi \left( \frac{t - u}{s} \right) \frac{ds}{s^2} \, du
\]

We are going to break up the scale integral into dyadic intervals. Define:

\[
\Delta_j(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} Wf(u, s) \frac{1}{\sqrt{s}} \psi \left( \frac{t - u}{s} \right) \frac{ds}{s^2} \, du, \quad j \in \mathbb{Z}
\]
Note that we have the following Littlewood-Paley type sum:

\[ f(t) = \sum_{j \in \mathbb{Z}} \Delta_j(t) \quad (37) \]

Let \( \Delta_j^{(k)}(t) \) be the \( k \)th order derivative of \( \Delta_j(t) \). To prove that \( f \) is Lipschitz \( \alpha \) at \( v \) we need a polynomial \( p_v(t) \) of degree \( \lfloor \alpha \rfloor \) and a constant \( K \) such that \( |f(t) - p_v(t)| \leq K|t - v|^{\alpha} \). We propose

\[ p_v(t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \left( \sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v) \right) \frac{(t - v)^k}{k!} \quad (38) \]

as a candidate. The remainder of the proof is in showing that (38) does the job. Notice that if \( f \) is \( n \) times differentiable at \( v \), then using (37) we have \( \sum_j \Delta_j^{(k)}(v) = f^{(k)}(v) \), and we get the jet \( J_v f(t) = p_v(t) \). However, we can compute \( \Delta_j^{(k)}(t) \) for each \( j \in \mathbb{Z} \) even when \( f \) is not \( n \) times differentiable at \( v \). But then we need to show that \( p_v(t) \) is well defined even when \( f \) is not \( n \) times differentiable at \( v \), and in particular we need to show that \( \sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v) \) is finite. Once we do that, we can think of (38) as a generalization of jets, and in particular the sums \( \sum_j \Delta_j^{(k)}(v) \) as generalizations of pointwise derivatives at \( v \).

We first prove that \( \sum_j \Delta_j^{(k)}(v) \) is finite by getting appropriate upper bounds on \( |\Delta_j^{(k)}(t)| \) for \( k \leq \lfloor \alpha \rfloor + 1 \leq n \). To simplify notation, we let \( K \) be a generic constant that may change from line to line, but does not depend on \( j \) and \( t \). Equation (36) plus the fast decay of the wavelet derivatives yield:

\[
|\Delta_j^{(k)}(t)| = \frac{1}{C_\psi} \left| \int_{-\infty}^{+\infty} \int_{2j}^{2j+1} Wf(u,s) \frac{d^k}{\sqrt{s} dt^k} \psi \left( \frac{t-u}{s} \right) \frac{ds}{s^2} du \right| \\
\leq \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2j}^{2j+1} |Wf(u,s)| \frac{1}{\sqrt{s} s^k} \left| \psi^{(k)} \left( \frac{t-u}{s} \right) \right| \frac{ds}{s^2} du \\
\leq K \int_{-\infty}^{+\infty} \int_{2j}^{2j+1} s^{\alpha-k} \left( 1 + \frac{|u-v|}{s} \right)^{\alpha'} \frac{C_m}{1 + |(t-u)/s|^m} \frac{ds}{s^2} du \quad (39)
\]

Now observe that on the interval \([2^j, 2^{j+1}]\), we have

\[ \sup_{s \in [2^j, 2^{j+1}]} s^{\alpha-k} = 2^{\alpha-k} 2^{j(\alpha-k)} = K 2^{j(\alpha-k)} \]
It follows that
\[(39) \leq K \int_{-\infty}^{+\infty} 2^{j(\alpha-k)} \left( 1 + \left| \frac{u-v}{2^j} \right|^{\alpha'} \right) \frac{1}{1 + \left| (t-u)/2^j \right|^m} \left( \int_{2^j}^{2^{j+1}} \frac{ds}{s^2} \right) \, du \]
\[\leq K \int_{-\infty}^{+\infty} 2^{j(\alpha-k)} \left( 1 + \left| \frac{u-v}{2^j} \right|^{\alpha'} \right) \frac{1}{1 + \left| (t-u)/2^j \right|^m} \, \frac{du}{2^j} \tag{40} \]

Now make the change of variables \( u' = 2^{-j}(u - t) \) and once again use the inequality \(|a + b|^{\alpha'} \leq 2^{\alpha'}(|a|^{\alpha'} + |b|^{\alpha'})\) to arrive at:
\[(40) = K 2^{j(\alpha-k)} \int_{-\infty}^{+\infty} \left( 1 + \left| u' + \frac{t-v}{2^j} \right|^{\alpha'} \right) \frac{1}{1 + |u'|^m} \, du' \]
\[\leq K 2^{j(\alpha-k)} \int_{-\infty}^{+\infty} \frac{1 + 2^{\alpha'}|u|^{\alpha'} + 2^{\alpha'}|t-v|/2^j |u|^{\alpha'}}{1 + |u'|^m} \, du' \]
\[\leq K 2^{j(\alpha-k)} \left[ \int_{-\infty}^{+\infty} \frac{1 + |u'|^{\alpha'}}{1 + |u'|^m} \, du' + \left| t-v \right|/2^j \left( \int_{-\infty}^{+\infty} \frac{1}{1 + |u'|^m} \, du' \right) \right] \]

Choosing \( m = \alpha' + 2 \) yields:
\[|\Delta_j^{(k)}(t)| \leq K 2^{j(\alpha-k)} \left( 1 + \left| \frac{t-v}{2^j} \right|^{\alpha'} \right), \quad \forall k \leq \lfloor \alpha \rfloor + 1 \tag{41} \]

At \( t = v \), we obtain
\[|\Delta_j^{(k)}(v)| \leq K 2^{j(\alpha-k)} \]
which guarantees a fast decay of of \(|\Delta_j^{(k)}(v)|\) as \( j \to -\infty \) (i.e., in the small scale regime), because \( \alpha \) is not an integer and so \( \alpha > \lfloor \alpha \rfloor \).

At large scales, since
\[|Wf(u, s)| = |f * \tilde{\psi}_s(u)| \leq \|f\|_2 \|\psi\|_2 \]
with the change variables \( u' = (t - u)/s \) we have
\[|\Delta_j^{(k)}(t)| \leq \frac{1}{C_{\psi}} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} |Wf(u, s)| \frac{1}{\sqrt{s}} \left| \frac{d^k}{dt^k} \psi \left( \frac{t-u}{s} \right) \right| \frac{ds}{s^2} \, du \]
\[\leq \frac{\|f\|_2 \|\psi\|_2}{C_{\psi}} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} \left| \psi^{(k)}(u') \right| \frac{ds}{s^{3/2+k}} \, du' \]
\[\leq K \frac{\|f\|_2 \|\psi\|_2 \|\psi^{(k)}\|_1}{C_{\psi}} 2^{-j(k+1/2)} \]

97
and therefore

\[ |\Delta_j^{(k)}(v)| \leq K2^{-j(k+1/2)} \]

Thus we can bound \( \sum_j \Delta_j^{(k)}(v) \) since

\[
\forall k \leq \lfloor \alpha \rfloor, \quad \left| \sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v) \right| \leq \sum_{j \in \mathbb{Z}} |\Delta_j^{(k)}(v)| \\
= \sum_{j=-\infty}^{0} |\Delta_j^{(k)}(v)| + \sum_{j=1}^{+\infty} |\Delta_j^{(k)}(v)| \\
\leq K \sum_{j=-\infty}^{0} 2^j(\alpha-k) + K \sum_{j=1}^{+\infty} 2^{-j(k+1/2)} \\
< +\infty
\]

With the Littlewood-Paley sum (37) we compute:

\[
|f(t) - p_v(t)| = \left| \sum_{j \in \mathbb{Z}} \left( \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right) \right| \quad (42)
\]

The sum over the scales \( j \in \mathbb{Z} \) is divided in two at \( J \) such that

\[ 2^{J-1} \leq |t - v| \leq 2^{J} \]

Notice the summands of (42) are \( \lfloor \alpha \rfloor \) Taylor approximations of \( \Delta_j(t) \) around \( v \). For the large scales corresponding to \( j \geq J \), we can use the classical Taylor’s theorem to get a bound:

\[
I = \sum_{j \geq J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\
\leq \sum_{j \geq J} \frac{|t - v|^{|\alpha|+1}}{(|\alpha|+1)!} \sup_{h \in [t,v]} |\Delta_j^{[\alpha]+1}(h)|
\]

98
Using $|t - v| \leq 2^J$ and inserting the bound (41) yields:

$$I \leq K|t - v|^{|\alpha|+1} \sum_{j \geq J} 2^{-j(|\alpha|+1-\alpha)} \left(1 + \frac{|t - v|^{|\alpha|}}{2^j}\right)$$

$$\leq K|t - v|^{|\alpha|+1} \sum_{j \geq J} 2^{-j(|\alpha|+1-\alpha)}$$

$$\leq K|t - v|^{|\alpha|+1}$$

$$= K|t - v|^{|\alpha|+1-\alpha}|t - v|^\alpha$$

$$\leq K2^{J(|\alpha|+1-\alpha)}|t - v|^\alpha$$

$$\leq K|t - v|^\alpha$$

Now consider the sum over $j < J$ and use (41),

$$II = \sum_{j < J} \left| \Delta_j(t) - \sum_{k=0}^{|\alpha|} \Delta_j^{(k)}(v) \frac{(t - v)^k}{k!} \right|$$

$$\leq K \sum_{j < J} \left[ 2^\alpha \left(1 + \frac{|t - v|^{|\alpha'|}}{2^j}\right) + \sum_{k=0}^{|\alpha|} \frac{|t - v|^k}{k!} 2^{j(|\alpha| - k)} \right]$$

$$\leq K \left[ 2^\alpha J + 2^{(\alpha - \alpha')J}|t - v|^\alpha' + \sum_{k=0}^{|\alpha|} \frac{|t - v|^k}{k!} 2^{J(|\alpha| - k)} \right]$$

Now use $2^{J-1} \leq |t - v| \leq 2^J$ to conclude that $II \leq K|t - v|^\alpha$. As a result,

$$|f(t) - p_v(t)| \leq I + II \leq K|t - v|^\alpha$$

which proves that $f$ is Lipschitz $\alpha$ at $v$. \qed
References


