We can use this link with Taylor's theorem to define notions of local regularity rather than global regularity. In particular, a function $f(t)$ is pointwise Lipschitz $\alpha \geq 0$ at $v \in \mathbb{R}$ if there exists $K > 0$ and a polynomial $p_v(t)$ of degree $n = \lfloor \alpha \rfloor$ such that

$$|f(t) - p_v(t)| \leq K|t - v|^{\alpha} \tag{30}$$

Furthermore, a function f is uniformly Lipschitz α over an interval $[a, b]$ if it satisfies (30) for all $v \in [a, b]$ with a constant K that is independent of v.

At each $v \in \mathbb{R}$ the polynomial $p_v(t)$ is unique. Additionally, if f is $n = \lfloor \alpha \rfloor$ times continuously differentiable in a neighborhood of v, then $p_v(t) = J_v f(t)$.

A function that is bounded but discontinuous at v is Lipschitz 0 at v. If $\alpha < 1$ at v, then f is not differentiable at v and α characterizes the type of singularity.

The next theorem generalizes Theorem 11 by relating the decay of the Fourier transform of $f(t)$ to the α regularity of f.

Theorem 29. *Suppose that $f \in \mathbf{L}^1(\mathbb{R})$. If*

$$\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|(1 + |\omega|^{\alpha})\, d\omega < +\infty \tag{31}$$

then $f \in \mathbf{C}^{\alpha}(\mathbb{R})$.

Proof. Equation (31) implies that $\widehat{f} \in \mathbf{L}^1(\mathbb{R})$, and so the Fourier inversion formula (2) holds. We use it to prove $f \in \mathbf{L}^{\infty}(\mathbb{R})$:

$$\begin{aligned} |f(t)| \leq \frac{1}{2\pi} \left| \int_{-\infty}^{+\infty} \widehat{f}(\omega) e^{i\omega t}\, d\omega \right| &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{f}(\omega)|\, d\omega \\ &\leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\widehat{f}(\omega)|(1 + |\omega|^{\alpha})\, d\omega < \infty \end{aligned}$$

Now suppose that $0 \leq \alpha < 1$ and show that $f \in \dot{\mathbf{C}}^{\alpha}(\mathbb{R})$. To do so we need to show there exists $K > 0$ such that

$$|f(t) - f(v)| \leq K|t - v|^{\alpha}, \quad \forall\, t, v \in \mathbb{R}$$

By the Fourier inversion formula (2) we have that

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \widehat{f}(\omega) e^{i\omega t}\, d\omega$$

It follows that

$$\frac{|f(t)-f(v)|}{|t-v|^\alpha} \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\widehat{f}(\omega)|\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha}\,d\omega$$

For $|\omega| \geq |t-v|^{-1}$,

$$\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha} \leq \frac{2}{|t-v|^\alpha} \leq 2|\omega|^\alpha \tag{32}$$

On the other hand, for $|\omega| \leq |t-v|^{-1}$, we note that if a function $h \in \dot{\mathbf{C}}^1(\mathbb{R})$, then

$$|h(t)-h(v)| \leq K|t-v|, \quad K = \sup_{u\in\mathbb{R}} |h'(u)|$$

Note that $e_\omega \in \mathbf{C}^1(\mathbb{R})$, where $e_\omega(t) = e^{i\omega t}$, and $|e'_\omega(t)| = |\omega|$. Therefore,

$$\frac{|e^{i\omega t}-e^{i\omega v}|}{|t-v|^\alpha} \leq \frac{|\omega||t-v|}{|t-v|^\alpha} = |\omega||t-v|^{1-\alpha} \leq |\omega||\omega|^{\alpha-1} = |\omega|^\alpha \tag{33}$$

Combining (32) and (33), we obtain

$$\frac{|f(t)-f(v)|}{|t-v|^\alpha} \leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2|\widehat{f}(\omega)||\omega|^\alpha\,d\omega = K$$

Equation (31) ensures that $K < \infty$, and so $f \in \mathbf{C}^\alpha(\mathbb{R})$.

We now extend the result to $\alpha \geq 1$. Let $n = \lfloor \alpha \rfloor$. Theorem 11 proves that $f \in \mathbf{C}^n(\mathbb{R})$. Recall that $\widehat{f^{(n)}}(\omega) = (i\omega)^n \widehat{f}(\omega)$. Equation (31) gives:

$$\int_{-\infty}^{+\infty} |\widehat{f^{(n)}}(\omega)|(1+|\omega|^{\alpha-n})\,d\omega = \int_{-\infty}^{+\infty} |\widehat{f}(\omega)|(|\omega|^n + |\omega|^\alpha)\,d\omega < \infty$$

Thus by our work above, we have that $f^{(n)} \in \mathbf{C}^{\alpha-n}(\mathbb{R})$, which proves that $f \in \mathbf{C}^\alpha(\mathbb{R})$. □

As we have discussed previously for $\mathbf{C}^n$-smooth functions, the decay of the Fourier transform can only indicate the minimum regularity of $f(t)$. Wavelet transforms characterize both the global and pointwise regularity of functions.

Exercise 50. Read Section 6.1.1 of *A Wavelet Tour of Signal Processing.*

Exercise 51. Consider the function

$$f(t) = t \sin\left(\frac{1}{t}\right)$$

(a) [5 points] Prove that $f(t)$ is pointwise Lipschitz 1 for all $t \in (-1, 1)$.

(b) [15 points] Prove that $f \in \mathbf{C}^\alpha(-1, 1)$ only for $\alpha \leq 1/2$ (*Hint:* Consider the points $t_n = (n + 1/2)^{-1}\pi^{-1}$).

Exercise 52. Show that f may be pointwise Lipschitz $\alpha > 1$ at v, while f' is not pointwise Lipschitz $\alpha - 1$ at v. Consider $f(t) = t^2 \cos(1/t)$ at $t = 0$.

Exercise 53. Find a function $f(t)$ for which there exists a constant K such that

$$|f(t) - f(v)| \leq K|u - v|, \quad \forall\, t, v \in \mathbb{R}$$

but does not satisfy (31) with $\alpha = 1$.

5.1.2 Wavelet Vanishing Moments

Section 6.1.2 of A Wavelet Tour of Signal Processing.

We assume throughout that $\psi(t)$ is a real valued wavelet.

A wavelet ψ has n vanishing moments if

$$\int_{-\infty}^{+\infty} t^k \psi(t)\, dt = 0, \quad \forall\, 0 \leq k < n$$

A wavelet ψ with n vanishing moments is orthogonal to polynomials of degree $n - 1$.

Suppose now that f is Lipschitz $\alpha < n$ at v, so that

$$f(t) = p_v(t) + \varepsilon_v(t)$$

with $p_v(t)$ a polynomial of degree $n - 1$ and

$$|\varepsilon_v(t)| \leq K|t - v|^\alpha$$

We have that

$$Wp_v(u, s) = \int_{-\infty}^{+\infty} p_v(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t - u}{s}\right) dt = \sqrt{s} \int_{-\infty}^{+\infty} p_v(st' + u) \psi(t')\, dt' = 0$$

Therefore,

$$Wf(u,s) = Wp_v(u,s) + W\varepsilon_v(u,s) = W\varepsilon_v(u,s)$$

Thus a wavelet transform with n vanishing moments analyzes $f(t)$ around $t = v$ by ignoring the polynomial approximation of $f(t)$ and focusing on the residual $\varepsilon_v(t)$.

A wavelet ψ has fast decay if

$$\forall\, m \in \mathbb{N},\ \ \exists\, C_m \text{ such that } |\psi(t)| \leq \frac{C_m}{1+|t|^m}, \quad \forall\, t \in \mathbb{R}$$

The following theorem shows that a wavelet ψ with fast decay and n vanishing moments is the n^{th} derivative of a function $\theta(t)$. The resulting wavelet transform is thus a multiscale differential operator.

Theorem 30. *A wavelet $\psi(t)$ with fast decay has n vanishing moments if and only if there exists $\theta(t)$ with a fast decay such that*

$$\psi(t) = (-1)^n \theta^{(n)}(t)$$

Consequently,

$$Wf(u,s) = s^n \frac{d^n}{du^n}(f * \widetilde{\theta}_s)(u)$$

where $\widetilde{\theta}_s(t) = s^{-1/2}\theta(-t/s)$. Furthermore, ψ has no more than n vanishing moments if and only if

$$\int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$$

Proof. Suppose that ψ has fast decay and n vanishing moments. Since ψ has fast decay we must have that $\widehat{\psi} \in \mathbf{C}^\infty(\mathbb{R})$; this follows from Theorem 11 by setting $f = \widehat{\psi}$. Thus we can differentiate $\widehat{\psi}(\omega)$ as many times as we like.

Recall that the Fourier transform of $h(t) = (-it)^k \psi(t)$ is $\widehat{h}(\omega) = \widehat{\psi}^{(k)}(\omega)$. It follows that

$$\widehat{\psi}^{(k)}(0) = \int_{-\infty}^{+\infty} (-it)^k \psi(t)\, dt = (-i)^k \int_{-\infty}^{+\infty} t^k \psi(t)\, dt = 0, \quad \forall\, 0 \leq k < n$$

We can therefore write $\widehat{\psi}$ as

$$\widehat{\psi}(\omega) = (-i\omega)^n \widehat{\theta}(\omega)$$

where $\widehat{\theta} \in \mathbf{L}^\infty$ since $\widehat{\psi} \in \mathbf{L}^\infty(\mathbb{R})$. It follows that

$$\psi(t) = (-1)^n \theta^{(n)}(t)$$

The fast decay of $\theta(t)$ is proved with an induction on n. For $n = 1$,

$$\widehat{\psi}(\omega) = -i\omega\widehat{\theta}(\omega) \Longrightarrow \psi(t) = -\theta'(t)$$

It follows that

$$\theta(t) = -\int_{-\infty}^{t} \psi(u)\, du$$

Thus, using the fast decay of $\psi(t)$,

$$|\theta(t)| \leq \int_{-\infty}^{t} |\psi(u)|\, du \leq \int_{-\infty}^{t} \frac{C_m}{1 + |u|^m}\, du \leq \frac{C'_{m-1}}{1 + |t|^{m-1}}, \quad \forall\, m \geq 2$$

Now make the inductive hypothesis that if $\Psi(t)$ is any wavelet with fast decay and

$$\widehat{\Psi}(\omega) = (-i\omega)^k \widehat{\Theta}(\omega), \quad 1 \leq k \leq n$$

then $\Theta(t)$ has fast decay. Consider now a wavelet ψ with fast decay that has $n+1$ vanishing moments, so that $\widehat{\psi}(\omega) = (-i\omega)^{n+1}\widehat{\theta}(\omega)$. Define

$$\widehat{\Theta}(\omega) = -i\omega\widehat{\theta}(\omega) \Longrightarrow \widehat{\psi}(\omega) = (-i\omega)^n \widehat{\Theta}(\omega)$$

By the inductive hypothesis, $\Theta(t)$ has fast decay. But then since $\widehat{\Theta}(\omega) = -i\omega\widehat{\theta}(\omega)$, we can apply the inductive hypothesis again to conclude that $\theta(t)$ has fast decay.

Conversely, suppose that $\psi(t) = (-1)^n \theta^{(n)}(t)$ and $\theta(t)$ has fast decay. Because of the fast decay,

$$|\widehat{\theta}(\omega)| \leq \int_{-\infty}^{+\infty} |\theta(t)|\, dt \leq \int_{-\infty}^{+\infty} \frac{C_m}{1 + |t|^m}\, dt < +\infty, \quad m \geq 2$$

Thus $\widehat{\theta} \in \mathbf{L}^\infty(\mathbb{R})$. The Fourier transform of $\psi(t)$ is

$$\widehat{\psi}(\omega) = (-i\omega)^n \widehat{\theta}(\omega)$$

It follows that $\widehat{\psi}^{(k)}(0) = 0$ for $k < n$. But then

$$\int_{-\infty}^{+\infty} t^k \psi(t)\, dt = i^k \widehat{\psi}^{(k)}(0) = 0, \quad 0 \leq k < n$$

Thus $\psi(t)$ has n vanishing moments.

To test whether $\psi(t)$ has more than n vanishing moments, we compute:

$$\int_{-\infty}^{+\infty} t^n \psi(t)\, dt = i^n \widehat{\psi}^{(n)}(0) = (-i)^n n! \widehat{\theta}(0)$$

Clearly then ψ has no more than n vanishing moments if and only if

$$\widehat{\theta}(0) = \int_{-\infty}^{+\infty} \theta(t)\, dt \neq 0$$

Recall the wavelet transform can be written as

$$Wf(u,s) = f * \widetilde{\psi}_s(u)$$

where

$$\widetilde{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi\left(\frac{-t}{s}\right) = \frac{(-1)^n}{\sqrt{s}} \theta^{(n)}\left(-\frac{t}{s}\right) = (-1)^n \widetilde{\theta}_s^{(n)}(t)$$

A simple calculation also shows that

$$\frac{d^n}{dt^n} \widetilde{\theta}_s(t) = \frac{1}{s^n} \frac{(-1)^n}{\sqrt{s}} \theta^{(n)}\left(-\frac{t}{s}\right) = \frac{(-1)^n}{s^n} \widetilde{\theta}_s^{(n)}(t) = \frac{\widetilde{\psi}_s(t)}{s^n}$$

Therefore $\widetilde{\psi}_s(t) = s^n (d^n/dt^n) \widetilde{\theta}_s(t)$. We then have:

$$Wf(u,s) = f * \widetilde{\psi}_s(u) = s^n f * \theta_s^{(n)}(u) = s^n \frac{d^n}{du^n} (f * \theta)(u)$$

□

If $K = \widehat{\theta}(0) \neq 0$, then the convolution $f * \widetilde{\theta}_s(t)$ can be interpreted as a weighted average of f with a kernel dilated by s. Theorem 30 proves that $Wf(u,s)$ is an n^{th} order derivative of an averaging of f over a domain proportional to s and centered at u. Figure plots $Wf(u,s)$ calculated with $\psi(t) = -\theta'(t)$, where $\theta(t)$ is a Gaussian. Notice how the sign and magnitude of the wavelet coefficients corresponds to the derivative of f averaged over a window of size proportional to s. Compare to Figure 19, which computed $Wf(u,s)$ with the Mexican hat wavelet $\psi(t) = \theta''(t)$ (θ again a Gaussian).

Since $\theta(t)$ has fast decay, once can verify that for any f that is continuous at u,

$$\lim_{s \to 0} f * \frac{1}{\sqrt{s}} \widetilde{\theta}_s(u) = K f(u)$$

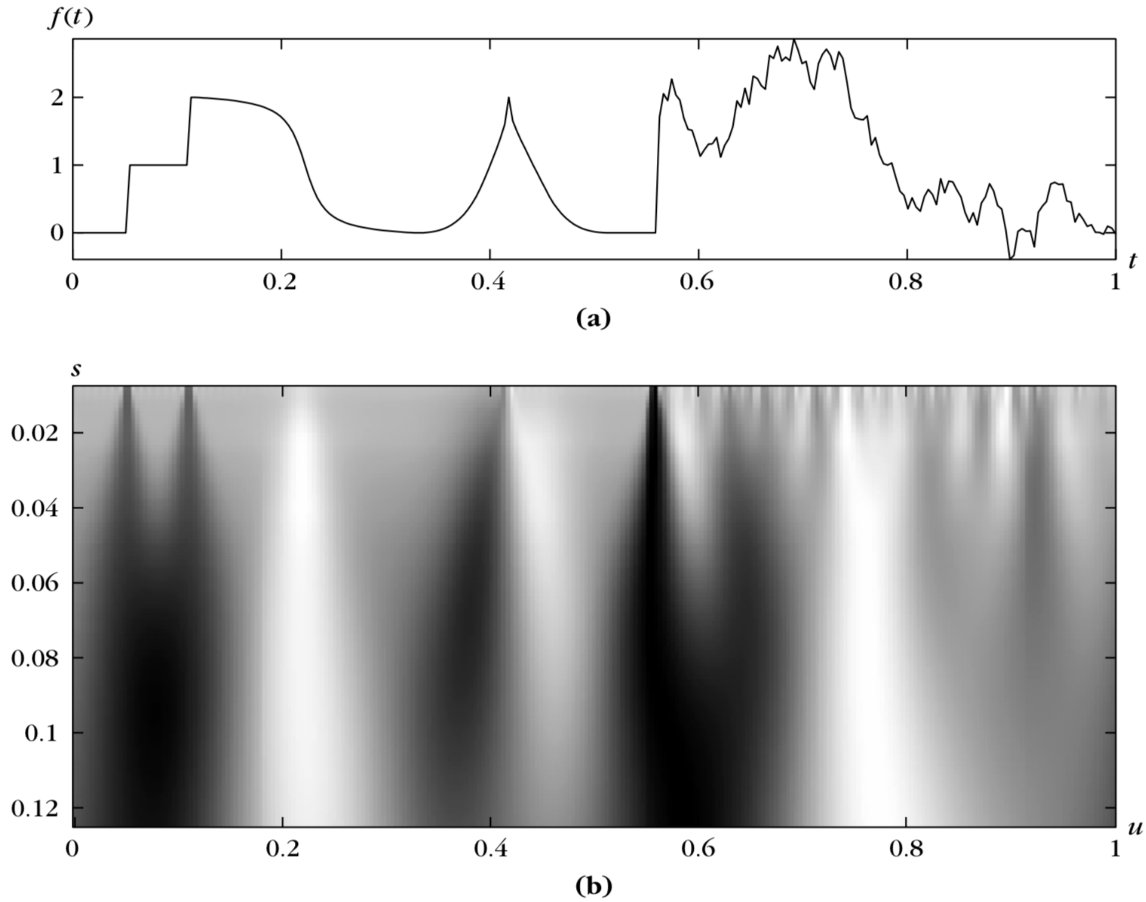

Figure 22: Wavelet transform $Wf(u,s)$ calculated with $\psi = -\theta'$, where θ is a Gaussian, for the signal $f(t)$ shown in (a). Position parameter u and scale s vary, respectively, along the horizontal and vertical axes. (b) Black, gray, and white points correspond to positive, zero, and negative wavelet coefficients. Singularities create large-amplitude coefficients in their cone of influence.

In the sense of distributions, we write

$$\lim_{s\to 0} \frac{1}{\sqrt{s}} \widetilde{\theta}_s(t) = K\delta(t)$$

If f is n times continuously differentiable in the neighborhood of u, then using Theorem 30,

$$\lim_{s\to 0} \frac{Wf(u,s)}{s^{n+1/2}} = \lim_{s\to 0} \frac{1}{\sqrt{s}} \frac{d^n}{dt^n}(f * \widetilde{\theta}_s)(u) = \lim_{s\to 0} f^{(n)} * \frac{1}{\sqrt{s}} \widetilde{\theta}_s(u) = Kf^{(n)}(u) \quad (34)$$

In particular, if $f \in \mathbf{C}^n(\mathbb{R})$, then $|Wf(u,s)| = O(s^{n+1/2})$. This gives us a first relation between the decay of $|Wf(u,s)|$ as $s \to 0$ and the uniform regularity of f. Next we push harder and obtain finer relations.

Exercise 54. Read Section 6.1.2 of *A Wavelet Tour of Signal Processing.*

Exercise 55. Let $f(t) = |t|^\alpha$. Show that $Wf(u,s) = s^{\alpha+1/2}Wf(u/s,1)$. Prove that it is not sufficient to measure the decay of $|Wf(u,s)|$ when $s \to 0$ at $u = 0$ in order to compute the Lipschitz regularity of f at $t = 0$.

5.1.3 Regularity Measurements with Wavelets

Section 6.1.3 of A Wavelet Tour of Signal Processing.

We now prove that "zooming in" on the wavelet coefficients of a signal f characterizes the pointwise regularity of f. We utilize a real valued wavelet $\psi \in \mathbf{C}^n(\mathbb{R})$ with n vanishing moments and with derivatives that have fast decay. The latter point means that for any $0 \leq k \leq n$ and $m \in \mathbb{N}$ there exists $C_m \geq 0$ such that

$$|\psi^{(k)}(t)| \leq \frac{C_m}{1+|t|^m}, \quad \forall\, t \in \mathbb{R}$$

Theorem 31. *If $f \in \mathbf{L}^2(\mathbb{R})$ is Lipschitz $\alpha \leq n$ at $v \in \mathbb{R}$, then there exits $A > 0$ such that*

$$|Wf(u,s)| \leq As^{\alpha+1/2}\left(1 + \left|\frac{u-v}{s}\right|^\alpha\right), \quad \forall\, (u,s) \in \mathbb{R} \times (0,\infty) \quad (35)$$

Conversely if $\alpha < n$ is not an integer and there exist A and $\alpha' < \alpha$ such that

$$|Wf(u,s)| \leq As^{\alpha+1/2}\left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right) \quad (36)$$

then $f(t)$ is Lipschitz α at $t = v$.

Proof. We first prove (35). Since f is Lipschitz α at v, there exists a polynomial $p_v(t)$ of degree $\lfloor \alpha \rfloor < n$ and K such that

$$|f(t) - p_v(t)| \leq K|t-v|^\alpha$$

Recall that since ψ has n vanishing moments, $Wp_v(u,s) = 0$. Therefore:

$$\begin{aligned} |Wf(u,s)| &= \left| \int_{-\infty}^{+\infty} [f(t) - p_v(t)] \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt \right| \\ &\leq \int_{-\infty}^{+\infty} K|t-v|^\alpha \frac{1}{\sqrt{s}} \left| \psi\left(\frac{t-u}{s}\right) \right| dt \end{aligned}$$

Now make the change of variables $x = (t-u)/s$, which induces the change of measure $dt = sdx$,

$$\begin{aligned} |Wf(u,s)| &\leq K\sqrt{s} \int_{-\infty}^{+\infty} |sx + u - v|^\alpha |\psi(x)| \, dx \\ &\leq K\sqrt{s} \int_{-\infty}^{+\infty} 2^\alpha (|sx|^\alpha + |u-v|^\alpha) |\psi(x)| \, dx \\ &= K 2^\alpha s^{\alpha + 1/2} \left(\int_{-\infty}^{+\infty} |x|^\alpha |\psi(x)| \, dx + \left| \frac{u-v}{s} \right|^\alpha \int_{-\infty}^{+\infty} |\psi(x)| \, dx \right) \\ &\leq A s^{\alpha+1/2} \left(1 + \left| \frac{u-v}{s} \right|^\alpha \right) \end{aligned}$$

where

$$A = K 2^\alpha \max \left(\int_{-\infty}^{+\infty} |x|^\alpha |\psi(x)| \, dx, \|\psi\|_1 \right)$$

and where we used the fact that $|a+b|^\alpha \leq 2^\alpha(|a|^\alpha + |b|^\alpha)$.

Now we prove (36). This is a difficult proof that adapts the Littlewood-Paley approach referenced earlier in the course. Recall the real wavelet inverse formula from Theorem 28,

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_0^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2} \, du$$

We are going to break up the scale integral into dyadic intervals. Define:

$$\Delta_j(t) = \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2} \, du, \quad j \in \mathbb{Z}$$

Note that we have the following Littlewood-Paley type sum:

$$f(t) = \sum_{j \in \mathbb{Z}} \Delta_j(t) \tag{37}$$

Let $\Delta_j^{(k)}(t)$ be the k^{th} order derivative of $\Delta_j(t)$. To prove that f is Lipschitz α at v we need a polynomial $p_v(t)$ of degree $\lfloor \alpha \rfloor$ and a constant K such that $|f(t) - p_v(t)| \leq K|t - v|^{\alpha}$. We propose

$$p_v(t) = \sum_{k=0}^{\lfloor \alpha \rfloor} \left(\sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v) \right) \frac{(t-v)^k}{k!} \tag{38}$$

as a candidate. The remainder of the proof is in showing that (38) does the job. Notice that if f is n times differentiable at v, then using (37) we have $\sum_j \Delta_j^{(k)}(v) = f^{(k)}(v)$, and we get the jet $J_v f(t) = p_v(t)$. However, we can compute $\Delta_j^{(k)}(t)$ for each $j \in \mathbb{Z}$ even when f is not n times differentiable at v. But then we need to show that $p_v(t)$ is well defined even when f is not n times differentiable at v, and in particular we need to show that $\sum_{j \in \mathbb{Z}} \Delta_j^{(k)}(v)$ is finite. Once we do that, we can think of (38) as a generalization of jets, and in particular the sums $\sum_j \Delta_j^{(k)}(v)$ as generalizations of pointwise derivatives at v.

We first prove that $\sum_j \Delta_j^{(k)}(v)$ is finite by getting appropriate upper bounds on $|\Delta_j^{(k)}(t)|$ for $k \leq \lfloor \alpha \rfloor + 1 \leq n$. To simplify notation, we let K be a generic constant that may change from line to line, but does not depend on j and t. Equation (36) plus the fast decay of the wavelet derivatives yield:

$$\begin{aligned}
|\Delta_j^{(k)}(t)| &= \frac{1}{C_\psi} \left| \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} Wf(u,s) \frac{1}{\sqrt{s}} \frac{d^k}{dt^k} \psi\left(\frac{t-u}{s}\right) \frac{ds}{s^2}\, du \right| \\
&\leq \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} |Wf(u,s)| \frac{1}{\sqrt{s}} \frac{1}{s^k} \left| \psi^{(k)}\left(\frac{t-u}{s}\right) \right| \frac{ds}{s^2}\, du \\
&\leq K \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} s^{\alpha - k} \left(1 + \left|\frac{u-v}{s}\right|^{\alpha'}\right) \frac{C_m}{1 + |(t-u)/s|^m} \frac{ds}{s^2}\, du
\end{aligned} \tag{39}$$

Now observe that on the interval $[2^j, 2^{j+1}]$, we have

$$\sup_{s \in [2^j, 2^{j+1}]} s^{\alpha - k} = 2^{\alpha - k} 2^{j(\alpha - k)} = K 2^{j(\alpha - k)}$$

It follows that

$$
\begin{aligned}
(39) &\leq K \int_{-\infty}^{+\infty} 2^{j(\alpha-k)} \left(1 + \left|\frac{u-v}{2^j}\right|^{\alpha'}\right) \frac{1}{1+|(t-u)/2^j|^m} \left(\int_{2^j}^{2^{j+1}} \frac{ds}{s^2}\right) du \\
&\leq K \int_{-\infty}^{+\infty} 2^{j(\alpha-k)} \left(1 + \left|\frac{u-v}{2^j}\right|^{\alpha'}\right) \frac{1}{1+|(t-u)/2^j|^m} \frac{du}{2^j} \qquad (40)
\end{aligned}
$$

Now make the change of variables $u' = 2^{-j}(u-t)$ and once again use the inequality $|a+b|^{\alpha'} \leq 2^{\alpha'}(|a|^{\alpha'} + |b|^{\alpha'})$ to arrive at:

$$
\begin{aligned}
(40) &= K2^{j(\alpha-k)} \int_{-\infty}^{+\infty} \left(1 + \left|u' + \frac{t-v}{2^j}\right|^{\alpha'}\right) \frac{1}{1+|u'|^m} \, du' \\
&\leq K2^{j(\alpha-k)} \int_{-\infty}^{+\infty} \frac{1 + 2^{\alpha'}|u|^{\alpha'} + 2^{\alpha'}|(t-v)/2^j|^{\alpha'}}{1+|u'|^m} \, du' \\
&\leq K2^{j(\alpha-k)} \left[\int_{-\infty}^{+\infty} \frac{1+|u'|^{\alpha'}}{1+|u'|^m} \, du' + \left|\frac{t-v}{2^j}\right|^{\alpha'} \int_{-\infty}^{+\infty} \frac{1}{1+|u'|^m} \, du'\right]
\end{aligned}
$$

Choosing $m = \alpha' + 2$ yields:

$$
|\Delta_j^{(k)}(t)| \leq K2^{j(\alpha-k)} \left(1 + \left|\frac{t-v}{2^j}\right|^{\alpha'}\right), \quad \forall\, k \leq \lfloor \alpha \rfloor + 1 \qquad (41)
$$

At $t = v$, we obtain

$$
|\Delta_j^{(k)}(v)| \leq K2^{j(\alpha-k)}
$$

which guarantees a fast decay of of $|\Delta_j^{(k)}(v)|$ as $j \to -\infty$ (i.e., in the small scale regime), because α is not an integer and so $\alpha > \lfloor \alpha \rfloor$.

At large scales, since

$$
|Wf(u,s)| = |f * \widetilde{\psi}_s(u)| \leq \|f\|_2 \|\psi\|_2
$$

with the change variables $u' = (t-u)/s$ we have

$$
\begin{aligned}
|\Delta_j^{(k)}(t)| &\leq \frac{1}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} |Wf(u,s)| \frac{1}{\sqrt{s}} \left|\frac{d^k}{dt^k}\psi\left(\frac{t-u}{s}\right)\right| \frac{ds}{s^2} \, du \\
&\leq \frac{\|f\|_2 \|\psi\|_2}{C_\psi} \int_{-\infty}^{+\infty} \int_{2^j}^{2^{j+1}} |\psi^{(k)}(u')| \, \frac{ds}{s^{3/2+k}} \, du' \\
&\leq \frac{K \|f\|_2 \|\psi\|_2 \|\psi^{(k)}\|_1}{C_\psi} 2^{-j(k+1/2)}
\end{aligned}
$$

and therefore

$$|\Delta_j^{(k)}(v)| \leq K 2^{-j(k+1/2)}$$

Thus we can bound $\sum_j \Delta_j^{(k)}(v)$ since

$$\begin{aligned}
\forall\, k \leq \lfloor \alpha \rfloor, \quad \left| \sum_{j\in\mathbb{Z}} \Delta_j^{(k)}(v) \right| &\leq \sum_{j\in\mathbb{Z}} |\Delta_j^{(k)}(v)| \\
&= \sum_{j=-\infty}^{0} |\Delta_j^{(k)}(v)| + \sum_{j=1}^{+\infty} |\Delta_j^{(k)}(v)| \\
&\leq K \sum_{j=-\infty}^{0} 2^{j(\alpha-k)} + K \sum_{j=1}^{+\infty} 2^{-j(k+1/2)} \\
&< +\infty
\end{aligned}$$

With the Littlewood-Paley sum (37) we compute:

$$|f(t) - p_v(t)| = \left| \sum_{j\in\mathbb{Z}} \left(\Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right) \right| \tag{42}$$

The sum over the scales $j \in \mathbb{Z}$ is divided in two at J such that

$$2^{J-1} \leq |t-v| \leq 2^J$$

Notice the summands of (42) are $\lfloor \alpha \rfloor$ Taylor approximations of $\Delta_j(t)$ around v. For the large scales corresponding to $j \geq J$, we can use the classical Taylor's theorem to get a bound:

$$\begin{aligned}
I &= \sum_{j\geq J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor \alpha \rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\
&\leq \sum_{j\geq J} \frac{|t-v|^{\lfloor \alpha \rfloor + 1}}{(\lfloor \alpha \rfloor + 1)!} \sup_{h\in[t,v]} |\Delta_j^{\lfloor \alpha \rfloor + 1}(h)|
\end{aligned}$$

Using $|t-v| \leq 2^J$ and inserting the bound (41) yields:

$$\begin{aligned}
I &\leq K|t-v|^{\lfloor\alpha\rfloor+1} \sum_{j \geq J} 2^{-j(\lfloor\alpha\rfloor+1-\alpha)} \left(1 + \left|\frac{t-v}{2^j}\right|^{\alpha'}\right) \\
&\leq K|t-v|^{\lfloor\alpha\rfloor+1} \sum_{j \geq J} 2^{-j(\lfloor\alpha\rfloor+1-\alpha)} \\
&\leq K|t-v|^{\lfloor\alpha\rfloor+1} \\
&= K|t-v|^{\lfloor\alpha\rfloor+1-\alpha}|t-v|^{\alpha} \\
&\leq K2^{J(\lfloor\alpha\rfloor+1-\alpha)}|t-v|^{\alpha} \\
&\leq K|t-v|^{\alpha}
\end{aligned}$$

Now consider the sum over $j < J$ and use (41),

$$\begin{aligned}
II &= \sum_{j<J} \left| \Delta_j(t) - \sum_{k=0}^{\lfloor\alpha\rfloor} \Delta_j^{(k)}(v) \frac{(t-v)^k}{k!} \right| \\
&\leq K \sum_{j<J} \left[2^{\alpha j} \left(1 + \left|\frac{t-v}{2^j}\right|^{\alpha'}\right) + \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{|t-v|^k}{k!} 2^{j(\alpha-k)} \right] \\
&\leq K \left[2^{\alpha J} + 2^{(\alpha-\alpha')J}|t-v|^{\alpha'} + \sum_{k=0}^{\lfloor\alpha\rfloor} \frac{|t-v|^k}{k!} 2^{J(\alpha-k)} \right]
\end{aligned}$$

Now use $2^{J-1} \leq |t-v| \leq 2^J$ to conclude that $II \leq K|t-v|^{\alpha}$. As a result,

$$|f(t) - p_v(t)| \leq I + II \leq K|t-v|^{\alpha}$$

which proves that f is Lipschitz α at v. □

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