The pseudo-inverse implements a signal synthesis with the *(canonical) dual frame*, defined by:

$$\widetilde{\phi}_\gamma = (\Phi^*\Phi)^{-1}\phi_\gamma$$

which has associated frame analysis operator

$$\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma \rangle$$

The next theorem shows that the dual frame synthesis operator is indeed the pseudo-inverse of the original frame analysis operator, and that the dual frame is in fact a frame.

Remark 40. Previously we used the notation $\widetilde{\phi}_\gamma(t) = \phi_\gamma^*(-t)$ for $\phi_\gamma \in \mathbf{L}^2(\mathbb{R})$. The dual frame definition has nothing to do with this definition. Moving forward we will let $\widetilde{\phi}_\gamma$ denote the dual frame vector, and we will use $\bar{\phi}_\gamma(t) = \phi_\gamma^*(-t)$ when $\phi_\gamma \in \mathbf{L}^2(\mathbb{R})$.

Theorem 41. *Let $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ be a frame with frame bounds $0 < A \leq B < \infty$. Then the dual frame synthesis operator satisfies*

$$\widetilde{\Phi}^* = \Phi^\dagger \tag{49}$$

and thus

$$f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma = \sum_{\gamma\in\Gamma} \langle f, \widetilde{\phi}_\gamma \rangle \phi_\gamma \tag{50}$$

Furthermore, the dual dictionary

$$\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$$

is a frame (hence the name dual frame) with frame bounds $0 < 1/B \leq 1/A < \infty$, meaning that

$$\frac{1}{B}\|f\|^2 \leq \sum_{\gamma\in\Gamma} |\langle f, \widetilde{\phi}_\gamma \rangle|^2 \leq \frac{1}{A}\|f\|^2, \quad \forall f \in \mathcal{H} \tag{51}$$

If the frame is tight (i.e., $A = B$), then

$$\widetilde{\phi}_\gamma = \frac{1}{A}\phi_\gamma$$

To prove this theorem, we will need the following lemma.

Lemma 42. *If $L : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator such that there exists $0 < A \leq B < \infty$ satisfying*

$$A\|f\|^2 \leq \langle Lf, f\rangle \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H} \tag{52}$$

then L is invertible and

$$\frac{1}{B}\|f\|^2 \leq \langle L^{-1}f, f\rangle \leq \frac{1}{A}\|f\|^2, \quad \forall\, f \in \mathcal{H} \tag{53}$$

Proof. Suppose first that $\mathcal{H}$ is finite dimensional of dimension N. Since L is self-adjoint, it has an orthonormal set of eigenvectors $e_1, \ldots, e_N \in \mathcal{H}$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ such that

$$Le_k = \lambda_k e_k, \quad \forall\, 1 \leq K \leq N$$

Equation (52) implies that $A \leq \lambda_k \leq B$ for each k. The operator L is therefore invertible, and it's eigenvalues are λ_k^{-1} with the same orthonormal eigenvectors e_k for $1 \leq k \leq N$. It follows that (53) must hold. The proof is extended to infinite dimensions using the same technique as in the proof of Theorem 37. □

Proof of Theorem 41. We first rewrite the dual analysis operator (noting that $\Phi^*\Phi$ is self-adjoint, and thus so is $(\Phi^*\Phi)^{-1}$):

$$\begin{aligned}\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma\rangle &= \langle f, (\Phi^*\Phi)^{-1}\phi_\gamma\rangle \\ &= \langle (\Phi^*\Phi)^{-1} f, \phi_\gamma\rangle \\ &= \Phi(\Phi^*\Phi)^{-1} f(\gamma)\end{aligned}$$

Thus

$$\widetilde{\Phi} = \Phi(\Phi^*\Phi)^{-1}$$

and we compute:

$$\widetilde{\Phi}^* = (\Phi^*\Phi)^{-1}\Phi^* = \Phi^\dagger$$

That proves (49).

Note that (50) can be written as:

$$I = \widetilde{\Phi}^*\Phi = \Phi^*\widetilde{\Phi}$$

where I is the identity operator. Since $\widetilde{\Phi}^* = \Phi^\dagger$, we have

$$\widetilde{\Phi}^*\Phi = \Phi^\dagger\Phi = I \tag{54}$$

Using the facts that $(\widetilde{\Phi}^*\Phi)^* = \Phi^*\widetilde{\Phi}$ and $I^* = I$, and taking the adjoint of both sides of (54), we obtain the second equality.

For the proof of (51), we use Lemma 42. Recall that the frame conditions can be rewritten as:

$$A\|f\|^2 \leq \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Applying Lemma 42 to $L = \Phi^*\Phi$ proves that

$$\frac{1}{B}\|f\|^2 \leq \langle (\Phi^*\Phi)^{-1} f, f\rangle \leq \frac{1}{A}\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Furthermore, using the first part of the proof we have:

$$\begin{aligned}\sum_{\gamma\in\Gamma} |\langle f, \widetilde{\phi}_\gamma\rangle|^2 &= \|\widetilde{\Phi} f\|^2 \\ &= \langle \Phi(\Phi^*\Phi)^{-1} f, \Phi(\Phi^*\Phi)^{-1} f\rangle \\ &= \langle \Phi^*\Phi(\Phi^*\Phi)^{-1} f, (\Phi^*\Phi)^{-1} f\rangle \\ &= \langle f, (\Phi^*\Phi)^{-1} f\rangle\end{aligned}$$

This proves (51).

If $A = B$, then

$$\langle \Phi^*\Phi f, f\rangle = A\|f\|^2, \quad \forall\, f \in \mathcal{H}$$

Thus the spectrum of $\Phi^*\Phi$ is only A, and we have $\Phi^*\Phi = AI$. It follows that $\widetilde{\phi}_\gamma = (\Phi^*\Phi)^{-1}\phi_\gamma = A^{-1}\phi_\gamma$. □

This theorem proves that one way to reconstruct a signal f from its frame coefficients $\Phi f(\gamma) = \langle f, \phi_\gamma\rangle$ is to use the dual frame $\widetilde{\phi}_\gamma$; equivalently, the synthesis coefficients of f in $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ are the dual frame coefficients $\widetilde{\Phi} f(\gamma) = \langle f, \widetilde{\phi}_\gamma\rangle$. If the frame is tight, then we have the simple reconstruction formula:

$$f = \frac{1}{A}\sum_{\gamma\in\Gamma} \langle f, \phi_\gamma\rangle \phi_\gamma$$

which mirrors the reconstruction of a signal f in an orthonormal basis, except for the factor of A^{-1}.

If $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ is a Riesz basis then the dictionary atoms are linearly independent, which implies that image$(\Phi) = \ell^2(\Gamma)$; therefore the dual frame $\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ is also a Riesz basis. Inserting $f = \phi_n$ into (50) yields:

$$\phi_n = \sum_{\gamma\in\Gamma} \langle \phi_n, \widetilde{\phi}_\gamma\rangle \phi_\gamma$$

The linear independence of $\mathcal{D}$ implies that the only expansion of ϕ_n is $\mathcal{D}$ is the trivial expansion $\phi_n = \phi_n$, which implies that

$$\langle \phi_n, \widetilde{\phi}_\gamma \rangle = \begin{cases} 1 & n = \gamma \\ 0 & n \neq \gamma \end{cases}$$

Thus the frame and dual frame are *biorthogonal bases* for $\mathcal{H}$. Furthermore, if the Riesz basis is normalized so that $\|\phi_\gamma\| = 1$ for all $\gamma \in \Gamma$, then using the dual frame bounds (51) and the biorthogonality we have:

$$\frac{1}{B} = \frac{1}{B}\|\phi_n\|^2 \leq \sum_{\gamma\in\Gamma} |\langle \phi_n, \widetilde{\phi}_\gamma \rangle|^2 = 1 \leq \frac{1}{A}\|\phi_n\|^2 = \frac{1}{A}$$

This shows that

$$A \leq 1 \leq B$$

Exercise 65. Read Section 5.1.2 of *A Wavelet Tour of Signal Processing.*

6.1.3 Dual Frame Analysis and Synthesis Computations

Section 5.1.3 of A Wavelet Tour of Signal Processing.

To compress and denoise a signal f we will project the signal onto a closed subspace $\mathbf{V} \subset \mathcal{H}$ that is generated from the span of a subset dictionary atoms from a larger dictionary. We thus need to study projections onto $\mathbf{V}$. As is well known from linear algebra, the best linear approximation of $f \in \mathcal{H}$ in $\mathbf{V}$ is the orthogonal projection of f onto $\mathbf{V}$. To make clear the setup, we let $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathcal{H}$ be a dictionary in $\mathcal{H}$, but which is a frame *only on* $\mathbf{V}$, i.e.,

$$A\|g\|^2 \leq \sum_{\gamma\in\Gamma} |\langle g, \phi_\gamma \rangle|^2 \leq B\|g\|^2, \quad \forall\, g \in \mathbf{V}$$

The analysis operator Φ is still defined on all of $\mathcal{H}$, but it may not behave "nicely" off of $\mathbf{V}$. The next theorem shows how to compute the orthogonal projection of $f \in \mathcal{H}$ onto $\mathbf{V}$ with the dual frame.

Theorem 43. *Let $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ be a frame for $\mathbf{V} \subset \mathcal{H}$, and $\widetilde{\mathcal{D}} = \{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ its dual frame in $\mathbf{V}$. The orthogonal projection of $f \in \mathcal{H}$ onto $\mathbf{V}$ is*

$$P_{\mathbf{V}} f = \sum_{\gamma\in\Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma = \sum_{\gamma\in\Gamma} \langle f, \widetilde{\phi}_\gamma \rangle \phi_\gamma \tag{55}$$

Proof. To show that $P_{\mathbf{V}}$ is a projection, we must show that $P_{\mathbf{V}}g = g$ for all $g \in \mathbf{V}$. But since $\mathcal{D}$ is a frame for $\mathbf{V}$, we have the synthesis formula given by (50) which proves that $P_{\mathbf{V}}g = g$ for all $g \in \mathbf{V}$.

To show that $P_{\mathbf{V}}$ is an orthogonal projection, we must verify that

$$\langle f - P_{\mathbf{V}}f, \phi_n \rangle = 0, \quad \forall\, n \in \Gamma$$

Note that (50) implies that

$$\phi_n = \sum_{\gamma \in \Gamma} \langle \phi_n, \widetilde{\phi}_\gamma \rangle \phi_\gamma$$

Therefore we compute:

$$\begin{aligned}
\langle f - P_{\mathbf{V}}f, \phi_n \rangle &= \langle f - \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma, \phi_n \rangle \\
&= \langle f, \phi_n \rangle - \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \langle \widetilde{\phi}_\gamma, \phi_n \rangle \\
&= \langle f, \phi_n - \sum_{\gamma \in \Gamma} \langle \widetilde{\phi}_\gamma, \phi_n \rangle^* \phi_\gamma \rangle \\
&= \langle f, \phi_n - \sum_{\gamma \in \Gamma} \langle \phi_n, \widetilde{\phi}_\gamma \rangle \phi_\gamma \rangle = 0
\end{aligned}$$

□

Since $\mathcal{D}$ is a frame for a subspace $\mathbf{V} \subset \mathcal{H}$, Φ is only invertible on this subspace and the definition of the pseudo-inverse is now:

$$\Phi^\dagger \Phi f = f, \ \forall\, f \in \mathbf{V} \quad \text{and} \quad \Phi^\dagger a = 0, \ \forall\, a \in \text{image}(\Phi)^\perp$$

Let $\Phi_{\mathbf{V}}$ be the restriction of the frame analysis operator to $\mathbf{V}$. The operator $\Phi^*\Phi_{\mathbf{V}}$ is invertible on $\mathbf{V}$ and we write $(\Phi^*\Phi_{\mathbf{V}})^{-1}$ as its inverse on $\mathbf{V}$. One can verify that

$$\Phi^\dagger = (\Phi^*\Phi_{\mathbf{V}})^{-1}\Phi^*$$

Let $f \in \mathcal{H}$. Theorem 43 and (55) give two ways in which to compute orthogonal projections onto $\mathbf{V}$. In a dual synthesis scenario, the orthogonal projection $P_{\mathbf{V}}f$ is computed from the frame analysis coefficients with the dual frame synthesis operator:

$$P_{\mathbf{V}}f = \widetilde{\Phi}^*\Phi f = \sum_{\gamma \in \Gamma} \langle f, \phi_\gamma \rangle \widetilde{\phi}_\gamma \tag{56}$$

If the frame $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ does not depend on the signal f, then the dual frame vectors are precomputed:

$$\widetilde{\phi}_\gamma = (\Phi^*\Phi_{\mathbf{V}})^{-1}\phi_\gamma$$

and the signal $P_{\mathbf{V}}f$ is synthesized with (56).

However, in many applications the frame vectors depend on the signal f. In this case the dual frame vectors $\widetilde{\phi}_\gamma$ cannot be computed in advance, and it is highly inefficient to compute them directly for each new signal f. In this case, we have already computed Φf and we want to compute $P_{\mathbf{V}}f$. We compute first:

$$y = \Phi^*\Phi f = \sum_{\gamma\in\Gamma}\langle f, \phi_\gamma\rangle\phi_\gamma \in \mathbf{V}$$

Let L be the linear operator defined as

$$Lh = \Phi^*\Phi_{\mathbf{V}}h, \quad \forall\, h \in \mathbf{V}$$

We then compute $P_{\mathbf{V}}f$ via:

$$P_{\mathbf{V}}f = L^{-1}y$$

We have already encountered several situations which would lead to something similar to the above scenario. For example, when we studied instantaneous frequencies we focused on the ridge points of either the windowed Fourier transform $Sf(u,\xi)$ or the wavelet transform $Wf(u,s)$. While these are not frames according to our current definition (since the index set (u,ξ) or (u,s) is uncountable), this is something we will remedy shortly. The subspace $\mathbf{V}$ then depends on the signal f since it is the subspace of $\mathcal{H}$ generated by the span of the $g_{u,\xi}$ or the $\psi_{u,s}$ that correspond to the ridge points of f in either the windowed Fourier or wavelet representation. Computing $P_{\mathbf{V}}f$ then synthesizes a signal $\widetilde{f}$ from only the ridge information of f. One can do something similar (and we will in a bit) when analyzing signals with isolated singularities and generating $\mathbf{V}$ as the span of the $\psi_{u,s}$ that correspond to the wavelet modulus maxima. As we shall see the synthesized signal $\widetilde{f} = P_{\mathbf{V}}f \approx f$, thus indicating that these local maxima points carry the majority of information in such signals.

The alternate scenario dual analysis, in which $P_{\mathbf{V}}f$ is computed as

$$P_{\mathbf{V}}f = \Phi^*\widetilde{\Phi}f = \sum_{\gamma\in\Gamma}\langle f, \widetilde{\phi}_\gamma\rangle\phi_\gamma$$

Similarly to before, if Φ does not depend upon f, then the dual frame vectors $\widetilde{\phi}_\gamma$ can be precomputed.

It is also possible in this case to view $\mathcal{D} = \{\phi_\gamma\}_{\gamma\in\Gamma}$ as a subset of a larger frame, which has been obtained by solving for a sparse approximation of f in the larger frame. We will come back to this idea later.

When $\mathcal{D}$ depends on f, we again go around computing the dual frame directly. Let

$$a[\gamma] = \widetilde{\Phi} f(\gamma)$$

and note that

$$P_{\mathbf{V}} f = \Phi^* a = \sum_{\gamma\in\Gamma} a[\gamma]\phi_\gamma$$

Since $\Phi P_{\mathbf{V}} f = \Phi f$, we have that

$$\Phi\Phi^* a = \Phi f$$

Let $\Phi^*_{\mathrm{Im}(\Phi)}$ be the restriction of Φ^* to image(Φ). Since $\Phi\Phi^*_{\mathrm{Im}(\Phi)}$ is invertible on image(Φ), we have

$$a = (\Phi\Phi^*_{\mathrm{Im}(\Phi)})^{-1}\Phi f$$

Notice that a is obtained by computing $a = L^{-1}y$, where in this case $y = \Phi f$ and $L = \Phi\Phi^*_{\mathrm{Im}(\Phi)}$.

Exercise 66. Read Section 5.1.3 of *A Wavelet Tour of Signal Processing*, including the algorithmic parts not covered above on how to compute the dual synthesis and dual analysis projections efficiently.

Exercise 67. Read Section 5.1.4 of *A Wavelet Tour of Signal Processing.*

6.1.4 Translation Invariant Frames

Section 5.1.5 of A Wavelet Tour of Signal Processing.

Let $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathbf{L}^2(\mathbb{R}^d)$ be a countable family of time frequency atoms. Recall that a translation invariant dictionary is a dictionary $\mathcal{D}$ of the form

$$\mathcal{D} = \{\phi_{u,\gamma}\}_{u\in\mathbb{R},\, \gamma\in\Gamma}$$

where

$$\phi_{u,\gamma}(x) = \phi_\gamma(x-u)$$

The analysis operator associated to $\mathcal{D}$ acts upon $f \in \mathbf{L}^2(\mathbb{R}^d)$ and is defined as

$$\Phi f(u,\gamma) = \langle f, \phi_{u,\gamma}\rangle = f * \bar{\phi}_\gamma(u), \quad \bar{\phi}_\gamma(x) = \phi_\gamma^*(-x)$$

Since the index set of $\mathcal{D}$ is $\mathbb{R}^d \times \Gamma$ is not countable, it is thus not strictly speaking a frame by the definition we have utilized up to this point. However, we can consider the energy of the transform $\Phi f(u,\gamma)$, which is defined as

$$\|\Phi f\|^2 = \sum_{\gamma\in\Gamma} \|\Phi f(\cdot,\gamma)\|_2^2 = \sum_{\gamma\in\Gamma} \int |\Phi f(u,\gamma)|^2\, du$$

If there exist $0 < A \leq B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{\gamma\in\Gamma} \|\Phi f(\cdot,\gamma)\|_2^2 = \sum_{\gamma\in\Gamma} \|f * \bar{\phi}_\gamma\|_2^2 \leq B\|f\|_2^2 \tag{57}$$

then all of the frame theory results we have studied thus far still apply. We will refer to such dictionaries as *semi-discrete frames*, since their index set is the cross product of $\mathbb{R}^d$ and Γ, where Γ is discrete but of course $\mathbb{R}^d$ is not. The next theorem shows that the semi-discrete frame condition (57) is equivalent to a condition on the Fourier transforms of the generators ϕ_γ

Theorem 44. *Let $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset \mathbf{L}^2(\mathbb{R}^d)$ be a family of generator functions. Then there exist $0 < A \leq B < \infty$ such that*

$$A \leq \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \leq B, \quad \textit{for almost every } \omega \in \mathbb{R}^d, \tag{58}$$

if and only if $\mathcal{D} = \{\phi_{u,\gamma}\}_{u\in\mathbb{R}^d,\gamma\in\Gamma}$ is a semi-discrete frame with frame bounds A and B. Any family $\{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ that satisfies

$$\sum_{\gamma\in\Gamma} \widehat{\phi}_\gamma^*(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = 1$$

defines a left inverse

$$f(x) = \sum_{\gamma\in\Gamma} \Phi f(\cdot,\gamma) * \widetilde{\phi}_\gamma(x)$$

and is thus the generators of the dual frame. They are defined in frequency as

$$\widehat{\widetilde{\phi}}_\gamma(\omega) = \frac{\widehat{\phi}_\gamma(\omega)}{\sum_{n\in\Gamma} |\widehat{\phi}_n(\omega)|^2}$$

Proof. Let $H : \mathbf{L}^2(\mathbb{R}^d) \to \mathbf{L}^2(\mathbb{R}^d)$ be defined as $Hf = f * \bar{h}$ for some filter h, where $\bar{h}(x) = h^*(-x)$. We first prove that $H^* g = g * h$. Indeed, using the Parseval formula (Theorem 6) and the convolution formula we have:

$$\begin{aligned}
\langle g, Hf \rangle &= \int_{\mathbb{R}^d} g(x)(f * \bar{h})^*(x)\, dx \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\omega) \widehat{f}^*(\omega) \widehat{\bar{h}}^*(\omega)\, d\omega \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{g}(\omega) \widehat{h}(\omega) \widehat{f}^*(\omega)\, d\omega \\
&= \int_{\mathbb{R}^d} g * h(x) f^*(x)\, dx \\
&= \langle g * h, f \rangle
\end{aligned}$$

Now assume that $\mathcal{D}$ is a semi-discrete frame with frame bounds A and B, and let $\Phi_\gamma f = \Phi f(\cdot, \gamma) = f * \bar{\phi}_\gamma$. Since $\mathcal{D}$ is a semi-discrete frame, each $\Phi_\gamma : \mathbf{L}^2(\mathbb{R}^d) \to \mathbf{L}^2(\mathbb{R}^d)$ and by the above computation $\Phi_\gamma^* g = g * \phi_\gamma$. The analysis operator is $\Phi : \mathbf{L}^2(\mathbb{R}^d) \to \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}^d))$ which can be written as $\Phi f = (\Phi_\gamma f)_{\gamma \in \Gamma}$. Let $G = (g_\gamma)_{\gamma \in \Gamma} \in \boldsymbol{\ell}^2(\mathbf{L}^2(\mathbb{R}^d))$ and now compute the adjoint of Φ:

$$\begin{aligned}
\langle G, \Phi f \rangle &= \sum_{\gamma \in \Gamma} \langle g_\gamma, \Phi_\gamma f \rangle \\
&= \sum_{\gamma \in \Gamma} \langle \Phi_\gamma^* g_\gamma, f \rangle \\
&= \left\langle \sum_{\gamma \in \Gamma} \Phi_\gamma^* g_\gamma, f \right\rangle \\
&= \left\langle \sum_{\gamma \in \Gamma} g_\gamma * \phi_\gamma, f \right\rangle
\end{aligned}$$

It follows that

$$\Phi^* G = \sum_{\gamma \in \Gamma} g_\gamma * \phi_\gamma$$

and furthermore

$$\Phi^* \Phi f = \sum_{\gamma \in \Gamma} f * \bar{\phi}_\gamma * \phi_\gamma$$

The semi-discrete frame condition (57) is equivalent to

$$A\|f\|^2 \leq \|\Phi f\|^2 = \langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$$

We can rewrite $\langle \Phi^*\Phi f, f\rangle$:

$$\begin{aligned}
\langle \Phi^*\Phi f, f\rangle &= \int_{\mathbb{R}^d} \sum_{\gamma\in\Gamma} f * \bar{\phi}_\gamma * \phi_\gamma(x) f^*(x)\, dx \\
&= \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^d} f * \bar{\phi}_\gamma * \phi_\gamma(x) f^*(x)\, dx \\
&= \sum_{\gamma} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\omega)\widehat{\phi}_\gamma^*(\omega)\widehat{\phi}_\gamma(\omega)\widehat{f}^*(\omega)\, d\omega \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \left(\sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\right) d\omega
\end{aligned}$$

Suppose by contradiction there exists $E \subset \mathbb{R}^d$ with finite but nonzero Lebesgue measure, i.e., $0 < |E| < \infty$, and for which

$$\sum_{\gamma} |\widehat{\phi}_\gamma(\omega)|^2 > B, \quad \forall\, \omega \in E$$

Let $\widehat{f}(\omega) = (2\pi)^{d/2}\mathbf{1}_E(\omega)$. We have that $\|\widehat{f}\|^2 = (2\pi)^d|E|$ and thus $f \in \mathbf{L}^2(\mathbb{R}^d)$ with $\|f\|^2 = |E|$. But then

$$\begin{aligned}
\langle \Phi^*\Phi f, f\rangle &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (2\pi)^d \mathbf{1}_E(\omega) \left(\sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\right) d\omega \\
&> B\int_E d\omega = B|E| = B\|f\|^2
\end{aligned}$$

which contradicts $\langle \Phi^*\Phi f, f\rangle \leq B\|f\|^2$. A similar argument proves the lower bound, and thus we have shown that

$$A \leq \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \leq B$$

Now assume that (58) holds. Let $f \in \mathbf{L}^2(\mathbb{R}^d)$ and multiply through by $(2\pi)^{-d}|\widehat{f}(\omega)|^2$ and integrate over $\mathbb{R}^d$ to obtain:

$$\frac{A}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2\, d\omega \leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2\, d\omega \leq \frac{B}{(2\pi)^d}\int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2\, d\omega$$

which is equivalent to

$$A\|f\|^2 \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \, d\omega \leq B\|f\|^2 \tag{59}$$

We rewrite the inner part:

$$\begin{aligned}
\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 \sum_{\gamma\in\Gamma} |\widehat{\phi}_\gamma(\omega)|^2 \, d\omega &= \sum_{\gamma\in\Gamma} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\omega)\widehat{\phi}^*_\gamma(\omega)\widehat{f}^*(\omega)\widehat{\phi}_\gamma(\omega) \, d\omega \\
&= \sum_{\gamma\in\Gamma} \int_{\mathbb{R}^d} f * \bar{\phi}_\gamma(x)(f * \bar{\phi}_\gamma)^*(x) \, dx \\
&= \sum_{\gamma\in\Gamma} \|\Phi f(\cdot,\gamma)\|^2
\end{aligned}$$

Plugging this into (59) proves that $\mathcal{D}$ is a semi-discrete frame.

Now let $\{\widetilde{\phi}_\gamma\}_{\gamma\in\Gamma}$ be a family of functions that satisfies

$$\sum_{\gamma\in\Gamma} \widehat{\phi}^*_\gamma(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = 1 \tag{60}$$

First, it is clear that such functions are defined in frequency as:

$$\widehat{\widetilde{\phi}}(\omega) = \frac{\widehat{\phi}_\gamma(\omega)}{\sum_{n\in\Gamma} |\widehat{\phi}_n(\omega)|^2} \tag{61}$$

by simply plugging (61) into the left hand side of (60) and verifying that the sum is equal to one. Now define

$$g(x) = \sum_{\gamma\in\Gamma} \Phi(\cdot,\gamma) * \widetilde{\phi}_\gamma(x) = \sum_{\gamma\in\Gamma} f * \bar{\phi}_\gamma * \widetilde{\phi}_\gamma(x)$$

The Fourier transform of g is:

$$\widehat{g}(\omega) = \sum_{\gamma\in\Gamma} \widehat{f}(\omega)\widehat{\phi}^*_\gamma(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = \widehat{f}(\omega) \sum_{\gamma\in\Gamma} \widehat{\phi}^*_\gamma(\omega)\widehat{\widetilde{\phi}}_\gamma(\omega) = \widehat{f}(\omega)$$

It follows that $g = f$, which completes the proof. □

Exercise 68. Read Section 5.1.5 of *A Wavelet Tour of Signal Processing.*

Exercise 69. Let $\phi_p \in \mathbb{R}^N$ be defined as:

$$\phi_p[n] = \delta[(n-p) \bmod N] - \delta[(n-p-1) \bmod N], \quad 0 \leq p < N$$

and define $\mathbf{V}$ as:

$$\mathbf{V} = \left\{ f \in \mathbb{R}^N : \sum_{n=0}^{N-1} f[n] = 0 \right\}$$

Prove that the dictionary $\mathcal{D} = \{\phi_p\}_{0 \leq p < N}$ is a translation invariant frame for $\mathbf{V}$. Compute the frame bounds. Is it a numerically stable frame when N is large?

Exercise 70. [40 points] The *Zak transform* maps any $f \in \mathbf{L}^2(\mathbb{R})$ to:

$$Zf(u,\xi) = \sum_{l \in \mathbb{Z}} e^{2\pi i l \xi} f(u-l)$$

(a) Prove that $Z : \mathbf{L}^2(\mathbb{R}) \rightarrow \mathbf{L}^2[0,1]^2$ is a unitary operator, i.e. show that

$$\int_{-\infty}^{+\infty} f(t) g^*(t)\, dt = \int_0^1 \int_0^1 Zf(u,\xi) Zg^*(u,\xi)\, du\, d\xi$$

One approach is the following: Let $g(t) = \mathbf{1}_{[0,1]}(t)$ and consider

$$\mathcal{B} = \{g_{n,k}\}_{(n,k) \in \mathbb{Z}^2}, \quad g_{n,k}(t) = g(t-n) e^{2\pi i k t}$$

Verify that $\mathcal{B}$ is an orthonormal basis for $\mathbf{L}^2(\mathbb{R})$, and then show that $\{Zg_{n,k}\}_{(n,k) \in \mathbb{Z}^2}$ is an orthonormal basis for $\mathbf{L}^2[0,1]^2$.

(b) Prove that the inverse Zak transform is defined by:

$$Z^{-1}h(u) = \int_0^1 h(u,\xi)\, d\xi, \quad \forall\, h \in \mathbf{L}^2[0,1]^2$$

(c) Now let $g \in \mathbf{L}^2(\mathbb{R})$ be arbitrary and consider

$$\mathcal{D} = \{g_{n,k}\}_{(n,k) \in \mathbb{Z}^2}, \quad g_{n,k}(t) = g(t-n) e^{2\pi i k t}$$

Prove that $\mathcal{D}$ is a frame for $\mathbf{L}^2(\mathbb{R})$ with frame bounds $0 < A \leq B < \infty$ if and only if

$$A \leq |Zg(u,\xi)|^2 \leq B, \quad \forall\, (u,\xi) \in [0,1]^2 \tag{62}$$

(d) Prove that if (62) holds, then the dual window $\widetilde{g}$ of the dual frame $\widetilde{D}$ is defined by

$$Z\widetilde{g}(u,\xi) = \frac{1}{Zg^*(u,\xi)}$$

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