Let $\mathcal{P} \subseteq \Gamma_\infty$. The scattering norm is computed as:

$$\|S_J[\mathcal{P}]f\|^2 = \sum_{p \in \mathcal{P}} \|S_J[p]f\|_2^2$$

The next theorem shows that the scattering transform is non-expansive.

Theorem 48. *[9, Proposition 2.5] The scattering transform is non-expansive:*

$$\|S_J[\Gamma_\infty]f - S_J[\Gamma_\infty]h\| \leq \|f - h\|_2$$

Proof. Using (72) and Proposition 47 we have:

$$\begin{aligned}\|U[\Gamma^m]f - U[\Gamma^m]h\|^2 &\geq \|U_J U[\Gamma^m]f - U_J U[\Gamma^m]h\|^2 \\ &= \|S_J[\Gamma^m]f - S_J[\Gamma^m]h\|^2 + \|U[\Gamma^{m+1}]f - U[\Gamma^{m+1}]h\|^2\end{aligned}$$

from which it follows

$$\|S_J[\Gamma^m]f - S_J[\Gamma^m]h\|^2 \leq \|U[\Gamma^m]f - U[\Gamma^m]h\|^2 - \|U[\Gamma^{m+1}]f - U[\Gamma^{m+1}]h\|^2$$

Recalling that $\Gamma_\infty = \cup_{m=0}^{\infty} \Gamma^m$, we have

$$\begin{aligned}\|S_J[\Gamma_\infty]f - S_J[\Gamma_\infty]h\|^2 &= \sum_{m=0}^{\infty} \|S_J[\Gamma^m]f - S_J[\Gamma^m]h\|^2 \\ &\leq \sum_{m=0}^{\infty} \|U[\Gamma^m]f - U[\Gamma^m]h\|^2 - \|U[\Gamma^{m+1}]f - U[\Gamma^{m+1}]h\|^2 \\ &= \|U[\Gamma^0]f - U[\Gamma^0]h\|^2 - \lim_{M\to\infty} \|U[\Gamma^M]f - U[\Gamma^M]h\|^2 \\ &\leq \|f - h\|_2^2\end{aligned}$$

□

The scattering transform thus define a representation $\Phi(f)$ that is $\mathbf{L}^2$ stable.

7.2.3 Diffeomorphism Stability for Band-limited Functions

General scattering transforms are stable to diffeomorphism actions if we restrict ourselves to band-limited functions. Doing so removes the difficulty of having arbitrarily high frequencies. We describe the results in this section.

Define the space of R-band-limited functions as:

$$\mathcal{H}_R = \{f \in \mathbf{L}^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subseteq [-R, R]\}$$

Theorem 49. *[10, Theorem 1] There exists a constant $C > 0$ such that for any displacement field $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla \tau\|_\infty \leq 1/2$ we have*

$$\|S_J[\Gamma_\infty]f - S_J[\Gamma_\infty]L_\tau f\| \leq CR\|f\|_2\|\tau\|_\infty, \quad \forall\, f \in \mathcal{H}_R$$

To prove Theorem 49 we shall make use of Schur's Lemma:

Lemma 50 (Schur's Lemma)**.** *Let $K : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$ be an integral operator of the form*

$$Kf(x) = \int_{-\infty}^{+\infty} k(x,u)f(u)\,du$$

Suppose that

$$\int_{-\infty}^{+\infty} |k(x,u)|\,dx \leq C_1 \quad \textit{and} \quad \int_{-\infty}^{+\infty} |k(x,u)|\,du \leq C_2$$

Then

$$\|K\| \leq \sqrt{C_1 C_2}$$

Proof. Let $f \in \mathbf{L}^2(\mathbb{R})$. We bound $\|Kf\|_2^2$ as:

$$\begin{aligned}
\|Kf\|_2^2 &= \int_{-\infty}^{+\infty} |Kf(x)|^2\,dx \\
&= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} f(x,u)f(u)\,du \right|^2 dx \\
&\leq \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |k(x,u)|^{1/2}[|k(x,u)|^{1/2}|f(u)|]\,du \right)^2 dx \\
&\leq \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |k(x,u)|\,du \right) \left(\int_{-\infty}^{+\infty} |k(x,u)||f(u)|^2\,du \right) dx \qquad (73) \\
&\leq C_1 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |k(x,u)||f(u)|^2\,du\,dx \\
&= C_1 \int_{-\infty}^{+\infty} |f(u)|^2 \int_{-\infty}^{+\infty} |k(x,u)|\,dx\,du \qquad (74) \\
&\leq C_1 C_2 \|f\|_2^2
\end{aligned}$$

where (73) follows from Hölder's inequality, and (74) follows from Tonelli's Theorem. It follows that $\|K\|^2 \leq C_1 C_2$. □

Proof of Theorem 49. We first show that $\|\nabla\tau\|_\infty \leq 1/2$ implies that $L_\tau f \in \mathbf{L}^2(\mathbb{R})$. Indeed, the norm of $L_\tau f$ is:

$$\|L_\tau f\|_2^2 = \int_{-\infty}^{+\infty} |f(x-\tau(x))|^2\,dx$$

Make the change of variables $u = x - \tau(x)$, which induces the following change of measure:

$$\frac{du}{dx} = |1-\tau'(x)| \geq 1 - |\tau'(x)| \geq 1 - \|\nabla\tau\|_\infty \geq 1/2$$

It follows that

$$\|L_\tau f\|_2^2 = \int_{-\infty}^{+\infty} |f(x-\tau(x))|^2\,dx \leq 2\int_{-\infty}^{+\infty} |f(u)|^2\,du = 2\|f\|_2^2$$

Now using Theorem 48, we have that:

$$\|S_J[\Gamma_\infty]f - S_J[\Gamma_\infty]L_\tau f\| \leq \|f - L_\tau f\|_2$$

Our previous example illustrating the instability of the Fourier modulus shows that $\|f - L_\tau f\|_2$ cannot be bounded in terms of the size of τ for general $f \in \mathbf{L}^2(\mathbb{R})$; however, we can bound it in terms of R and $\|\tau\|_\infty$ when we restrict to $f \in \mathcal{H}_R$. The remainder of the proof shows how.

Recall the space of Schwartz class functions, denoted $\mathcal{S}(\mathbb{R})$. Functions $\eta \in \mathcal{S}(\mathbb{R})$ are infinitely differentiable with $\eta^{(k)}(x)$ rapidly decreasing for all $k \geq 0$, that is

$$\sup_{x\in\mathbb{R}} |x|^m|\eta^{(k)}(x)| < \infty, \quad \forall\, m,k \geq 0$$

It is a fact (that we shall not prove), that one can construct an $\eta \in \mathcal{S}(\mathbb{R})$ such that

$$\widehat{\eta}(\omega) = 1, \quad \forall\,\omega \in [-1,1]$$

It follows that if we set

$$\eta_R(x) = R\eta(Rx)$$

then

$$\widehat{\eta}_R(\omega) = \widehat{\eta}(\omega/R)$$

which implies that

$$\widehat{\eta}_R(\omega) = 1, \quad \forall\,\omega \in [-R,R]$$

In particular, for any $f \in \mathcal{H}_R$, we have

$$\widehat{f}(\omega) = \widehat{f}(\omega)\widehat{\eta}_R(\omega) \Longrightarrow f = f * \eta_R$$

As such we define the operator $A_{\eta_R} : \mathbf{L}^2(\mathbb{R}) \to \mathbf{L}^2(\mathbb{R})$ as

$$A_{\eta_R} f = f * \eta_R$$

We then have for all $f \in \mathcal{H}_R$,

$$\|f - L_\tau f\|_2 = \|A_{\eta_R} f - L_\tau A_{\eta_R} f\|_2 \leq \|A_{\eta_R} - L_\tau A_{\eta_R}\| \|f\|_2$$

We have thus reduced the problem to bounding the operator norm $\|A_{\eta_R} - L_\tau A_{\eta_R}\|$.

To bound this operator norm we shall use Schur's Lemma. Note that

$$\begin{aligned} L_\tau A_{\eta_R} f(x) - A_{\eta_R} f(x) &= \int_{-\infty}^{+\infty} f(u)\eta_R(x - \tau(x) - u)\, du - \int_{-\infty}^{+\infty} f(u)\eta_R(x - u)\, du \\ &= \int_{-\infty}^{+\infty} [\eta_R(x - \tau(x) - u) - \eta_R(x - u)] f(u)\, du \end{aligned}$$

Thus the kernel of the operator $L_\tau A_{\eta_R} - A_{\eta_R}$ is

$$k(x, u) = \eta_R(x - \tau(x) - u) - \eta_R(x - u)$$

Now we bound $|k(x, u)|$. Define:

$$h_{x,u}(t) = \eta_R(x - t\tau(x) - u) - \eta_R(x - u)$$

and note that

$$h_{x,u}(0) = 0 \qquad \text{and} \qquad h_{x,u}(1) = k(x, u)$$

Then using the Fundamental Theorem of Calculus,

$$h_{x,u}(t) = \int_0^t h'_{x,u}(\lambda)\, d\lambda$$

from which it follows:

$$|k(x, u)| = |h_{x,u}(1)| \leq \int_0^1 \left|h'_{x,u}(\lambda)\right|\, d\lambda$$

Thus we have $|h'_{x,u}(\lambda)|$:

$$\begin{aligned}|h'_{x,u}(\lambda)| &= \left|\frac{d\eta_R}{d\lambda}(x-\lambda\tau(x)-u)\right| \\ &= R|\eta'_R(x-\lambda\tau(x)-u)\tau(x)| \\ &\leq R\|\tau\|_\infty|\eta'_R(x-\lambda\tau(x)-u)|\end{aligned}$$

At last we bound the two one sided $\mathbf{L}^1$ integrals of $k(x,u)$. For the first we have (using Tonelli):

$$\begin{aligned}\int_{-\infty}^{+\infty}|k(x,u)|\,du &\leq R\|\tau\|_\infty\int_{-\infty}^{+\infty}\int_0^1|\eta'_R(x-\lambda\tau(x)-u)|\,d\lambda\,du \\ &= R\|\tau\|_\infty\int_0^1\int_{-\infty}^{+\infty}|\eta'_R(x-\lambda\tau(x)-u)|\,du\,d\lambda \\ &= R\|\tau\|_\infty\|\eta'_R\|_1 \\ &= R\|\eta'\|_1\|\tau\|_\infty\end{aligned}$$

For the other $\mathbf{L}^1$ integral we have:

$$\begin{aligned}\int_{-\infty}^{+\infty}|k(x,u)|\,dx &\leq R\|\tau\|_\infty\int_{-\infty}^{+\infty}\int_0^1|\eta'_R(x-\lambda\tau(x)-u)|\,d\lambda\,dx \\ &= R\|\tau\|_\infty\int_0^1\int_{-\infty}^{+\infty}|\eta'_R(x-\lambda\tau(x)-u)|\,dx\,d\lambda\end{aligned}$$

Now make the change of variables $y=x-\lambda\tau(x)-u$ which induces the change of measure (recall $0\leq\lambda\leq 1$):

$$\frac{dy}{dx}=|1-\lambda\tau'(x)|\geq 1-\lambda|\tau'(x)|\geq 1-\lambda\|\nabla\tau\|_\infty\geq 1-\lambda/2\geq 1/2$$

Thus we have:

$$\int_{-\infty}^{+\infty}|k(x,u)|\,dx\leq 2R\|\tau\|_\infty\int_0^1\int_{-\infty}^{+\infty}|\eta'_R(y)|\,dy\,d\lambda=2R\|\eta'\|_1\|\tau\|_\infty$$

Applying Schur's lemma with $C_1=R\|\eta'\|_1\|\tau\|_\infty$ and $C_2=2R\|\eta'\|_1\|\tau\|_\infty$ yields:

$$\|A_{\eta_R}-L_\tau A_{\eta_R}\|\leq\sqrt{2}\|\eta'\|_1R\|\tau\|_\infty$$

We conclude that:

$$\|S_J[\Gamma_\infty]f - S_J[\Gamma_\infty]L_\tau f\| \leq \sqrt{2}\|\eta'\|_1 R\|f\|_2\|\tau\|_\infty$$

□

7.2.4 Wavelet scattering and translation invariance

In order to obtain stable representations of non band-limited functions we replace the generic high pass filters with multiscale wavelet filters. In particular let ψ be a complex analytic wavelet, such as a Gabor type wavelet, and consider the dictionary of generators

$$\mathcal{D} = \{\phi_J\} \cup \{\psi_j\}_{j<J}$$

where $\psi_j(x) = 2^{-j}\psi(2^{-j}x)$. Scattering transforms are defined in the same way as above, but with an index set given by $\Gamma_J = \{j \in \mathbb{Z} : j < J\}$. To clarify, a path $p \in \Gamma_{J,\infty}$ is either $p = \emptyset$ or p is of the form $p = (j_1, \ldots, j_m)$ for any $m \geq 1$ with

$$S_J[p]f = |||f * \psi_{j_1}| * \psi_{j_2}| * \cdots * \psi_{j_m}| * \phi_J$$

The following theorem shows that the wavelet scattering transform satisfies a type of translation invariance property.

Theorem 51. *[9, Theorem 2.10] Let $c \in \mathbb{R}$ and $L_c f(x) = f(x-c)$ be the translation of f by c. Then there exists a constant $C > 0$ such that for any path set $\mathcal{P}_J \subseteq \Gamma_{J,\infty}$,*

$$\|S_J[\mathcal{P}_J]f - S_J[\mathcal{P}_J]L_c f\| \leq C2^{-J}|c|\|U[\mathcal{P}_J]f\|$$

Proof. Since translations commute with convolution operators and the complex modulus operator, it follows that

$$S_J[\mathcal{P}_J]L_c f = L_c S_J[\mathcal{P}_J]f$$

Thus:

$$\begin{aligned}\|S_J[\mathcal{P}_J]f - S_J[\mathcal{P}_J]L_c f\| &= \|S_J[\mathcal{P}_J]f - L_c S_J[\mathcal{P}_J]f\| \\ &= \|A_J U[\mathcal{P}_J]f - L_c A_J U[\mathcal{P}_J]f\| \\ &\leq \|A_J - L_c A_J\|\|U[\mathcal{P}_J]f\|\end{aligned}$$

Lemma 52 below proves there exists a constant $C > 0$ such that for any displacement field $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla\tau\|_\infty \leq 1/2$ we have

$$\|A_J - L_\tau A_J\| \leq C2^{-J}\|\tau\|_\infty$$

Applying the lemma to $\tau = c$ for which $\|\tau\|_\infty = |c|$ and $\|\nabla\tau\|_\infty = 0$ yields:

$$\|A_J - L_c A_J\| \leq C2^{-J}|c|$$

which completes the proof. □

Lemma 52. *[9, Lemma 2.11] There exists a constant $C > 0$ such that for all $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla\tau\|_\infty \leq 1/2$ we have*

$$\|A_J - L_\tau A_J\| \leq C2^{-J}\|\tau\|_\infty$$

Proof. This proof will use Schur's Lemma 50 as well and is similar to the proof of Theorem 49. In particular, the kernel of $L_\tau A_J - A_J$ is:

$$k_J(x,u) = \phi_J(x - \tau(x) - u) - \phi_J(x-u)$$

Using the same Fundamental Theorem of Calculus argument yields:

$$|k_J(x,u)| \leq 2^{-J}\|\tau\|_\infty \int_0^1 |\phi_J'(x - u - t\tau(x))|\,dt$$

Note that ϕ_J replaces η_R and 2^{-J} replaces R from the proof of Theorem 49. The same techniques as in that proof show similarly here that

$$\int_{-\infty}^{+\infty} |k_J(x,u)|\,du \leq 2^{-J}\|\tau\|_\infty\|\phi'\|_1 \quad \text{and} \quad \int_{-\infty}^{+\infty} |k_J(x,u)\,dx \leq 2^{-(J-1)}\|\tau\|_\infty\|\phi'\|_1$$

□

Theorem 51 seems to indicate that if we let $J \to \infty$ then we obtain a translation invariant representation of f. However, the right hand upper bound also depends upon J through the term $\|U[\mathcal{P}_J]f\|$, and so the issue is subtle. In fact one can prove that

$$\lim_{J\to\infty} \|S_J[\mathcal{P}_J]f - S_J[\mathcal{P}_J]L_c f\| = 0$$

but the proof is difficult and requires additional assumptions on the wavelet ψ; we refer the reader to [9, Theorem 2.10] for more details.

If we restrict to scattering paths of length no more than M though, we can more easily obtain a translation invariance result. Let $\Gamma_{J,M}$ denote all such paths:

$$\Gamma_{J,M} = \bigcup_{m=0}^{M} \Gamma_J^m$$

As a corollary to Theorem 51 we have:

Corollary 53. *Let $c \in \mathbb{R}$ and $L_c f(x) = f(x-c)$ be the translation of f by c. Then there exists a constant $C > 0$ such that for any $M \geq 0$,*

$$\|S_J[\Gamma_{J,M}]f - S_J[\Gamma_{J,M}]L_c f\| \leq C 2^{-J}\sqrt{M+1}|c|\|f\|_2$$

Proof. The result follows from Theorem 51 by taking $\mathcal{P}_J = \Gamma_{J,M}$ and then showing that

$$\|U[\Gamma_{J,M}]f\| \leq (M+1)\|f\|_2$$

We now prove this new inequality. We first show that

$$\|U[\Gamma_J^m]f\| \leq \|f\|_2 \tag{75}$$

Equation (75) clearly holds for $m = 0$ since $U[\Gamma_J^0]f = f$. Now let $k \geq 0$ and suppose by the inductive hypothesis that (75) holds for all $m \leq k$. Let us show it holds for $m = k+1$:

$$\begin{aligned}
\|U[\Gamma_J^{k+1}]f\|^2 &= \sum_{(j_1,\ldots,j_{k+1})\in\Gamma_J^{k+1}} \|U[j_{k+1}]U[j_k]\cdot U[j_1]f\|_2^2 \\
&= \sum_{(j_1,\ldots,j_k)\in\Gamma_J^k} \sum_{j_{k+1}\in\Gamma_J} \|U[j_k]\cdots U[j_1]f * \psi_{j_{k+1}}\|_2^2 \\
&\leq \sum_{(j_1,\ldots,j_k)\in\Gamma_J^k} \|U[j_k]\cdots U[j_1]f\|_2^2 \\
&= \|U[\Gamma_J^k]f\|^2 \leq \|f\|_2^2
\end{aligned}$$

We now observe that:

$$\|U[\Gamma_{J,M}]f\|^2 = \sum_{m=0}^{M} \|U[\Gamma_J^m]f\|^2 \leq (M+1)\|f\|_2^2$$

□

Corollary 53 shows that a scattering transform computed with paths of length no more than M provides a locally translation invariant metric up to the scale 2^J. Furthermore, the limit representation $\Phi(f) = \lim_{J\to\infty} S_J[\Gamma_{J,M}]f$ is translation invariant.

7.2.5 Wavelet scattering and diffeomorphism stability

Wavelet scattering transforms are stable to small diffeomorphism actions on a subclass of $\mathbf{L}^2(\mathbb{R})$ functions that includes functions with arbitrarily high frequency content. Scattering transforms over $\Gamma_{J,M}$ are stable to diffeomorphism actions for all $\mathbf{L}^2(\mathbb{R})$ functions. We state the main results and sketch the proof. To get started, define the mixed $(\boldsymbol{\ell}^1, \mathbf{L}^2(\mathbb{R}))$ norm of the propagator as:

$$\|U[\Gamma_{J,M}]f\|_1 = \sum_{m=0}^{M} \|U[\Gamma_J^m]f\|, \quad M \in \mathbb{N} \cup \{\infty\}$$

Theorem 54. *[9, Theorem 2.12] There exists a constant $C > 0$ such that for all $M \in \mathbb{N} \cup \{\infty\}$, all $f \in \mathbf{L}^2(\mathbb{R})$ with $\|U[\Gamma_{J,\infty}]f\|_1 < \infty$ and all $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla\tau\|_\infty \leq 1/2$ satisfy*

$$\|S_J[\Gamma_{J,M}]f - S_J[\Gamma_{J,M}]L_\tau f\| \leq C\|U[\Gamma_{J,M}]f\|_1 K(\tau) \tag{76}$$

where

$$K(\tau) = 2^{-J}\|\tau\|_\infty + \max(J,1)\|\nabla\tau\|_\infty + \|H\tau\|_\infty$$

Furthermore, for all $M \in \mathbb{N}$, all $f \in \mathbf{L}^2(\mathbb{R})$ and all $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla\tau\|_\infty <$ 1/2,

$$\|S_J[\Gamma_{J,M}]f - S_J[\Gamma_{J,M}]L_\tau f\| \leq C(M+1)\|f\|_2 K(\tau) \tag{77}$$

Proof. We first remark that (77) follows from (76) since by the same argument as in the proof of Corollary 53 we have:

$$\|U[\Gamma_{J,M}]f\|_1 = \sum_{m=0}^{M} \|U[\Gamma_J^m]f\| \leq \sum_{m=0}^{M} \|f\|_2 = (M+1)\|f\|_2$$

Now for the proof of (76). Define the commutator of two operators A and B as:

$$[A, B] = AB - BA$$

We have:

$$\|S_J[\Gamma_{J,M}]f - S_J[\Gamma_{J,M}]L_\tau f\| \leq \|S_J[\Gamma_{J,M}]f - L_\tau S_J[\Gamma_{J,M}]f\| + \|[S_J[\Gamma_{J,M}], L_\tau]f\|$$

We will bound the two right hand side terms separately.

The proof for the first term is familiar. As in the proof of Theorem 51, we have

$$\|S_J[\Gamma_{J,M}]f - L_\tau S_J[\Gamma_{J,M}]f\| \leq \|A_J - L_\tau A_J\| \|U[\Gamma_{J,M}]f\|$$

Lemma 52 proves that

$$\|A_J - L_\tau A_J\| \leq C2^{-J}\|\tau\|_\infty$$

Additionally, noting that $\|a\|_2 \leq \|a\|_1$ for $a \in \boldsymbol{\ell}^1$, we have

$$\|U[\Gamma_{J,M}]f\| = \left(\sum_{m=0}^{M} \|U[\Gamma_J^m]f\|^2\right)^{\frac{1}{2}} \leq \sum_{m=0}^{M} \|U[\Gamma_J^m]f\| = \|U[\Gamma_{J,M}]f\|_1$$

Therefore:

$$\|S_J[\Gamma_{J,M}]f - L_\tau S_J[\Gamma_{J,M}]f\| \leq C2^{-J}\|\tau\|_\infty \|U[\Gamma_{J,M}]f\|_1$$

We now turn to the second term, $\|[S_J[\Gamma_{J,M}], L_\tau]f\|$. Since $S_J[\Gamma_{J,M}]$ is obtained by iterating upon the non-expansive operator U_J, one can prove the following upper bound on scattering commutators:

Lemma 55. *[9, Lemma 2.13] For any operator L on $\mathbf{L}^2(\mathbb{R})$,*

$$\|[S_J[\Gamma_{J,M}], L]f\| \leq \|U[\Gamma_{J,M}]f\|_1 \|[U_J, L]\|$$

We omit the proof of the lemma and continue with the proof of the main theorem. Applying the lemma to $L = L_\tau$ we obtain

$$\|[S_J[\Gamma_{J,M}], L_\tau]f\| \leq \|U[\Gamma_{J,M}]f\|_1 \|[U_J, L_\tau]\|$$

So now we focus on $\|[U_J, L_\tau]\|$.

For the wavelet transform write the frame analysis operator as $\Phi_J = W_J$, where

$$W_J = \{A_J, W[j]\}_{j<J}, \quad A_J f = f * \phi_J \text{ and } W[j]f = f * \psi_j$$

Recall that $U_J = \{A_J, \mathcal{M}W[j]\}$, where $\mathcal{M}$ is the modulus operator, i.e., $\mathcal{M}(f) = |f|$. The commutator $[U_J, L_\tau]$ is thus:

$$\begin{aligned}
[U_J, L_\tau] &= U_J L_\tau - L_\tau U_J \\
&= \{A_J L_\tau, \mathcal{M}W[j]L_\tau\}_{j<J} - \{L_\tau A_J, L_\tau \mathcal{M}W[j]\}_{j<J} \\
&= \{A_J L_\tau - L_\tau A_J, \mathcal{M}W[j]L_\tau - \mathcal{M}L_\tau W[j]\}_{j<J} \\
&= \{[A_J, L_\tau], \mathcal{M}[W[j], L_\tau]\}_{j<J}
\end{aligned}$$

Since the modulus is non-expansive, it follows that

$$\|[U_J, L_\tau]\| \leq \|[W_J, L_\tau]\|$$

We have thus reduced the problem to bounding $\|[W_J, L_\tau]\|$, which is:

$$\|[W_J, L_\tau]f\|^2 = \|[A_J, L_\tau]f\|^2 + \sum_{j<J} \|[W[j], L_\tau]f\|^2$$

This is the primary difficulty of the proof. The following lemma completes the proof of the main theorem.

Lemma 56. *[9, Lemma 2.14] There exists $C > 0$ such that for all $J \in \mathbb{Z}$ and all $\tau \in \mathbf{C}^2(\mathbb{R})$ with $\|\nabla\tau\|_\infty \leq 1/2$ satisfy*

$$\|[W_J, L_\tau]\| \leq C[\max(J, 1)\|\nabla\tau\|_\infty + \|H\tau\|_\infty]$$

□

We do not prove Lemma 56 here, as the proof is long, difficult and technical. We describe the first part, though, to illustrate the main idea and its connection to other proofs we have carried out. Note that

$$\|[W_J, L_\tau]\| = \|[W_J, L_\tau]^*[W_J, L_\tau]\|^{1/2}$$

and that

$$[W_J, L_\tau]^*[W_J, L_\tau] = [A_J, L_\tau]^*[A_J, L_\tau] + \sum_{j<J}[W[j], L_\tau]^*[W[j], L_\tau]$$

Therefore:

$$\|[W_J, L_\tau]\| \leq \|[A_J, L_\tau]^*[A_J, L_\tau]\|^{1/2} + \left\|\sum_{j<J}[W[j], L_\tau]^*[W[j], L_\tau]\right\|^{1/2}$$

Part of the proof shows that

$$\|[A_J, L_\tau]^*[A_J, L_\tau]\|^{1/2} = \|[A_J, L_\tau]\| \leq C\|\nabla\tau\|_\infty$$

The majority of the proof is dedicated to proving that

$$\left\|\sum_{j<J}[W[j], L_\tau]^*[W[j], L_\tau]\right\|^{1/2} \leq C[\max(J, 1)\|\nabla\tau\|_\infty + \|H\tau\|_\infty]$$

To prove the second bound, factorize $[W[j], L_\tau]$ as:

$$[W[j], L_\tau] = K_j L_\tau$$

with

$$K_j = W[j] - L_\tau W[j] L_\tau^{-1}$$

Observe that

$$\left\| \sum_{j<J} [W[j], L_\tau]^* [W[j], L_\tau] \right\|^{1/2} \leq \|L_\tau\| \left\| \sum_{j<J} K_j^* K_j \right\|^{1/2}$$

with $\|L_\tau\| \leq (1 - \|\nabla\tau\|_\infty)^{-1} \leq 2$. The sum over $j < J$ is broken into low frequencies and high frequencies:

$$\left\| \sum_{j<J} K_j^* K_j \right\|^{1/2} \leq \left\| \sum_{0 \leq j < J} K_j^* K_j \right\|^{1/2} + \left\| \sum_{j<0} K_j^* K_j \right\|^{1/2}$$

The low frequency sum is bounded as

$$\left\| \sum_{0<j<J} K_j^* K_j \right\|^{1/2} \leq CJ \|\nabla\tau\|_\infty$$

and the high frequency term is bounded as:

$$\left\| \sum_{j<0} K_j^* K_j \right\|^{1/2} \leq C(\|\nabla\tau\|_\infty + \|H\tau\|_\infty)$$

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